Math 833 — Week 2

Appell sequences of real-rooted polynomials.

Our task for today is to motivate and prove:

Theorem 1: Let $p_k(x) = x^k + \dots$, $k = 0, 1, 2, \dots$ be an Appell sequence of polynomials: $p_{k-1}(x) = \frac{1}{k} \frac{\partial}{\partial x} p_k(x)$

Then all polynomials $p_k(x)$ are **real-rooted** (all their zeros are real)

if and only if

$\exists\, \gamma_1 \in \mathbb{R}$, $\gamma_2 \geq 0$ and sequence $(\alpha_i)_{i=1}^{\infty} \in \mathbb{R}^{\infty}$ with $\sum_{i=1}^{\infty} |\alpha_i|^2 < \infty$:

$$p_k(x) = \sum_{m=0}^{k} c_{k-m} \frac{k!}{m!} x^m = c_0 x^k + c_1 k x^{k-1} + c_2 k(k-1) x^{k-2} + \dots + c_k k!$$

for all $n = 0, 1, 2, \dots$, where (c_m) is found from

$$\sum_{m=0}^{\infty} c_m z^m = \exp\left(-\gamma_1 z - \frac{\gamma_2}{2} z^2\right) \cdot \prod_{i \geq 1} (1 - \alpha_i z) e^{\alpha_i z} \underset{\text{definition}}{=} \varphi_{\alpha, \gamma}(z)$$

Example 1: $\gamma_1 = 1$; $\gamma_2 = \delta_i = 0$.

$$\sum c_m z^m = e^{-z} \Rightarrow c_m = (-1)^m \cdot \frac{1}{m!}$$

$$p_k(x) = \sum_{m=0}^{k} (-1)^{k-m} \frac{k!}{m!\,(k-m)!} x^m = (x-1)^k \quad , k=0,1,2,..$$

Indeed, this is an Appell real-rooted sequence

Exercise 1: Compute $p_k(x)$ for the case

$\gamma_1 = \delta_1 = 1$, all other parameters $=0$. Check that you get an Appell sequence.

Exercise 2: Compute $p_k(x)$ for the case

$\gamma_1 = 2$, $\delta_1 = \delta_2 = 1$, all other parameters $=0$. Check that you get an Appell sequence.

Example 2: We take $\gamma_2 = 1$, all other parameters $= 0$.

$$\sum_{m \geq 0} c_m z^m = \exp\left(-\frac{z^2}{2}\right) . \Rightarrow \quad c_m = \begin{cases} 0, & m \text{ is odd}, \\ \dfrac{(-1)^{m/2}}{2^{m/2}\left(\frac{m}{2}\right)!}, & m \text{ is even}. \end{cases}$$

$$P_K(x) = \sum_{\substack{0 \leq m \leq K \\ K-m \text{ is even}}} (-1)^{\frac{K-m}{2}} \frac{K!}{(K-m)!\, m!\, 2^{(K-m)/2}} \cdot x^m$$

Comparing with [Week 1, Slide 9], we conclude that these are **Hermite polynomials**.

Corollary: Hermite polynomials $H_K(x)$ are real-rooted and satisfy $H_{K-1}(x) = \frac{1}{K} \frac{\partial}{\partial x} H_K(x)$

[Follows from Theorem 1 and Example 2, but there are also direct proofs in the orthogonal polynomials literature]

History of Theorem 1:

Journal für die reine und angewandte Mathematik | Volume 1914: Issue 144

- First appearance over 100 years ago:

Über zwei Arten von Faktorenfolgen in der Theorie der algebraischen Gleichungen.

Von Herrn *G. Pólya* in Budapest und Herrn *J. Schur* in Bonn.

At that point mathematicians were very much interested in locations of zeros of polynomials and more general functions and their behavior under various algebraic operations.

- Recently rediscovered in the context of general β random matrix theory (is coming later in this class)

Mathematics > Probability

[Submitted on 21 May 2019 (v1), last revised 20 Aug 2020 (this version, v2)]

The boundary of the orbital beta process

Theodoros Assiotis, Joseph Najnudel

Parameters α_i, γ_1, γ_2 in theorem 1 might look mysterious. However, they have an asymptotic interpretation.

Theorem 2 Let $p_N(x) = \prod_{i=1}^{} \left(x - a_i^N\right)$ be the Appell sequence corresponding to $\varphi_{\alpha,\gamma}$ in theorem 1.

Assume that both a_i^N an α_i are ordered by decreasing magnitude ($i=1$ corresponds to the ones with largest $|\cdot|$)

Then:

$$\alpha_i = \lim_{N\to\infty} \frac{a_i^N}{N}, \quad i = 1,2,..$$

existence of limits is a part of theorem 2

$$\gamma_1 = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{a_i^N}{N} \qquad [\text{"mean"}]$$

$$\gamma_2 = \lim_{N\to\infty} \sum_{i=1}^{N} \left[\left(\frac{a_i^N}{N}\right)^2 - \alpha_i^2\right] \qquad [\text{"variance"}]$$

Note that is is possible to have

$\gamma_1 \neq \sum \alpha_i$

and

$\gamma_2 \neq 0$!

Proof of Theorems 1 and 2. [you can skip the proof in the first reading]

Step 1: Take an arbitrary Appell sequence of real-rooted polynomials $P_N(x) = \prod_{i=1}^{N} (x - a_i^N)$.

We aim to show that the three limits in theorem 2 exist and, hence, we can use them to **define** $(\delta_i, \gamma_1, \gamma_2)$

Notation: $e_m^{(N)} = e_m(a_1^N, \dots, a_N^N) = \sum_{i_1 < \dots < i_m} a_{i_1}^N \cdot \dots \cdot a_{i_m}^N$

("elementary symmetric functions")

Vieta's formulas say that:

$$(*) \qquad P_N(x) = x^N - e_1^{(N)} x^{N-1} + e_2^{(N)} x^{N-2} - \dots + (-1)^m e_m^{(N)} x^{N-m} + \dots$$

Now choose $K < N$. Differentiating (*) we get

$$P_K(x) = \left(\frac{K!}{N!}\right)\cdot\left[\frac{N!}{K!}x^K - \frac{(N-1)!}{(K-1)!}e_1^{(N)}x^{K-1} + \dots + (-1)^m\frac{(N-m)!}{(K-m)!}e_m^{(N)}x^{K-m} + \dots\right]$$

normalization making leading coef. of $P_K(x)$ equal to 1

$$= x^K - \frac{K}{N}e_1^{(N)}x^{K-1} + \dots + (-1)^m\frac{K(K-1)\dots(K-m+1)}{N(N-1)\dots(N-m+1)}e_m^{(N)}x^{K-m}$$

Now fix K and start growing N. $P_K(x)$ does not change! Hence, for each $m=1,2,\dots$

$$\frac{e_m^{(N)}}{N(N-1)\dots(N-m+1)}$$ does not depend on N

Sending $N\to\infty$, we conclude that there exists a limit:

$$c_m := \lim_{N\to\infty}(-1)^m\frac{e_m^{(N)}}{N^m}$$

Let us restate that in the language of generating functions.

$$\sum_{m\geq 0} c_m z^m = \sum_{m\geq 0} z^m \lim_{N\to\infty} \sum_{i_1<\dots<i_m} \left(-\frac{a_{i_1}^N}{N}\right)\cdot\dots\cdot\left(-\frac{a_{i_m}^N}{N}\right) =$$

$$= \lim_{N\to\infty} \prod_{i=1}^{N}\left(1 - z\frac{a_i^N}{N}\right) \left[= \lim_{N\to\infty} \left(\frac{z}{N}\right)^N \cdot P_N\left(\frac{N}{z}\right)\right]$$

↑ term-wise limit of a series

Taking logarithm, we can rewrite this as

$$\lim_{N\to\infty} \exp\left(\sum_{i=1}^{N} \ln\left(1 - z\frac{a_i^N}{N}\right)\right) =$$

$$= \exp\left(-z\sum_{i=1}^{N}\frac{a_i^N}{N} - \frac{z^2}{2}\sum\left(\frac{a_i^N}{N}\right)^2 - \frac{z^3}{3}\sum\left(\frac{a_i^N}{N}\right)^3 - \dots\right)$$

Conclusion: Each power sum $\sum_{i=1}^{N}\left(\frac{a_i^N}{N}\right)^m$ should converge as $N\to\infty$.

Lemma 1: Each power sum $\sum_{i=1}^{N} \left(\frac{a_i^N}{N}\right)^m$ converges as $N \to \infty$ to a finite number p_m if and only if (**)

$$\exists \lim_{N\to\infty} \frac{a_i^N}{N} = \delta_i, \quad \exists \lim_{N\to\infty} \sum_{i=1}^{N} \frac{a_i}{N} = \gamma_1, \quad \exists \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{a_i}{N}\right)^2 - \sum_{i=1}^{\infty} \delta_i^2 = \gamma_2$$

and we have $p_1 = \gamma_1$, $p_2 = \sum_{i=1}^{\infty} \delta_i^2 + \gamma_2$, $p_m = \sum_{i=1}^{\infty} (\delta_i)^m$, $m > 2$

Proof of Lemma: First, assume that 3 limits (**) exist. Then:

- $\sum_i \left(\frac{a_i^N}{N}\right) \longrightarrow \gamma_1 = p_1$ directly by the 2nd limit
- $\sum_i \left(\frac{a_i^N}{N}\right)^2 \longrightarrow \sum_{i=1}^{\infty} \delta_i^2 + \gamma_2$ directly by the 2nd and 3rd limits
- $\sum_i \left(\frac{a_i^N}{N}\right)^m \longrightarrow \sum_i (\delta_i)^m$ because each individual term a_i/N converges towards a_i and we have a tail bound

$$\left| \sum_{i>I} \left(\frac{a_i^N}{N}\right)^m \right| < \left|\frac{a_I^N}{N}\right|^{m-2} \sum_{i>I} \left(\frac{a_i^N}{N}\right)^2 \longrightarrow 0 \text{ as } I \to \infty \text{ by 1st and 2nd limits}$$

In the opposite direction, suppose that all power sums converge. Hence, each sequence

$$\left(\frac{a_i^N}{N}\right), \quad N=1,2,\ldots$$ is bounded.

Therefore, we can pass to a subsequence N_k, such that all $\frac{a_i^{N_k}}{N_k}$ converge to some numbers d_i

Then we can use the first part of the proof to link d_i to limits of power sums. We conclude that the limits d_i are uniquely fixed by limits of power sums. Therefore, all subsequential limits of $\left(\frac{a_i^N}{N}\right)$ are the same and, hence, the sequence $\left(\frac{a_i^N}{N}\right)$, $N=1,2,\ldots$ converges, which finishes the proof.

[some details of this argument are missing and I leave it to the students to fill them in]

Back to the proof of theorems. Using Lemma 1, we get the desired existence of the limits

$$\alpha_i = \lim_{N\to\infty} \frac{a_i^N}{N}, \quad \gamma_1 = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{a_i^N}{N}, \quad \gamma_2 = \lim_{N\to\infty} \sum_{i=1}^{N} \left[\left(\frac{a_i^N}{N}\right)^2 - \alpha_i^2 \right].$$

Step 2 We now establish the formula for $\rho_n(x)$ corresponding to $(\alpha_i, \gamma_1, \gamma_2)$ as in Theorem 1.

Continuing slide 8:

$$\sum_{m\geq 0} C_m \cdot z^m = \lim_{N\to\infty} \exp\left(-z \sum \frac{a_i^N}{N} - \frac{z^2}{2} \sum \left(\frac{a_i}{N}\right)^2 - \dots\right) =$$

$$= \exp\left(-z\gamma_1 - \frac{z^2}{2}\left(\gamma_2 + \sum_i \alpha_i^2\right) - \frac{z^3}{3}\left(\sum \alpha_i^3\right) - \dots\right) =$$

$$= \exp\left(-z\gamma_1 - \frac{z^2}{2}\gamma_2\right) \cdot \exp\left(\sum_i \left[\ln(1-z\alpha_i) + z\alpha_i\right]\right) = \varphi_{\alpha,\gamma}(z)$$

Hence, continuing slide 7, we have

$$p_K(x) = x^K - K c_1 x^{K-1} + K(K-1)\cdot c_2 \cdot x^{K-2} - \dots$$

which matches the formula of theorem 1.

Step 3. The only thing which remains to prove is that if we construct polynomials $p_K(x)$ according to the formulas of theorem 1, then they form an Appell sequence (simple exercise from definition) and they are all real-rooted (needs an argument).

For that choose a sequence $p^{(N)}(x) = \prod_{i=1}^{N} (x - a_i^{(N)})$
[It does not have to be an Appel sequence]
so that the 3 limits of theorem 2 exist and give the desired $(d_i, \gamma_1, \gamma_2)$

Lemma 2: If $P(x)$ is a real-rooted polynomial, then $\frac{\partial}{\partial x} P(x)$ is also real-rooted.

Proof: Suppose that $P(x)$ has $N = \deg(P(x))$ real roots. Then for each pair of adjacent roots $a < b$ $P'(x)$ has a root on segment (a, b) — in the point of maximum/minimum of $P(x)$ on the segment. Hence, $P'(x)$ has $N-1$ real roots. Since its degree is $N-1$, all roots are real.

[Some additional details are needed in the case when some of the roots of $P(x)$ coincide — left to the students]

Now define $P_k(x) = \lim_{N\to\infty} \frac{k!}{N!} \left(\frac{\partial}{\partial x}\right)^{N-k} p^{(N)}(x)$

- By continuity of differentiation, $P_k(x)$ form an Appell sequence: $P_{k-1}(x) = \frac{1}{k} \frac{\partial}{\partial x} P_k(x)$
- By Lemma 2 and continuity of the property, $P_k(x)$ are real-rooted.
- By Step 2 of the proof polynomials $P_k(x)$ match those of theorem 1 corresponding to $\Phi_{\alpha,\gamma}(z)$.

Theorems 1 and 2 are proven!

Summary: The proof is based on the following scheme

- Aproximate infinite system (Appell sequence) by derivatives of degree N polynomial

- Investigate how parameters of this polynomial (roots a_i^N) should vary as $N \to \infty$ in order to have a well-defined limit

- Identify the limit and get a „nice" formula for it

The scheme of proof of a classification theorem through approximations will be repeated many times in this class. It is called „ergodic method".

Analysis of the answer in Theorem 1:

$$\Phi_{\alpha,\gamma} = \exp\left(-\gamma_1 z - \gamma_2 \frac{z^2}{2}\right) \prod_{i\geq 1} (1-\alpha_i z) e^{\alpha_i z}$$

Is constructed from 3 basic blocks

$$\Phi = \exp(-z) \qquad \Phi = \exp\left(-\frac{z^2}{2}\right) \qquad \Phi = (1-z)e^z$$

[We investigated corresponding polynomials on slides 2-3]
by rescaling of z-variable and multiplications.

Question: What is the meaning of these two operations in terms of polynomials?

The first one is elementary:

Theorem 3: Suppose that $P_k(x)$ is an Appell real-rooted sequence corresponding to $\Phi(z)$.

Then $c^k P_k\left(\frac{x}{c}\right)$ is the sequence corresponding to $\Phi(cz)$

Proof: Clearly, $c^k P_k\left(\frac{x}{c}\right) = c^k\left(\left(\frac{x}{c}\right)^k + \ldots\right) = x^k + \ldots$

are monic. Further, they are real-rooted (roots are multiplied by c). They are also Appell:

$$\frac{1}{k}\frac{\partial}{\partial x} c^k P_k\left(\frac{x}{c}\right) = c^k \cdot \frac{1}{c}\frac{1}{k} P_k'\left(\frac{x}{c}\right) = c^{k-1} P_{k-1}\left(\frac{x}{c}\right)$$

Finally, the series expansion coefficients of $\Phi(cz)$ are related to those of $\Phi(z)$ via:

$$C_k^{\Phi(cz)} = c^k \cdot C_k^{\Phi(z)}$$

Comparing with Theorem 1, we see the same transformation in $P_k \to c^k P_k\left(\frac{x}{c}\right)$

Multiplication of $\mathcal{P}(z)$ is a more complicated operation.

Definition: Finite free convolution is a binary operation on polynomials. Given

$$P_N(x) = \prod_{i=1}^{N} (x - a_i) \quad \text{and} \quad \widetilde{P}_N(x) = \prod_{i=1}^{N} (x - b_i)$$

in outputs $\left(P_N \oplus \widetilde{P}_N\right)(x) = \prod_{i=1}^{N} (x - c_i)$, such that

$$\prod_{i=1}^{N} (x - c_i) = \frac{1}{N!} \sum_{\sigma \in S_N} \prod_{i=1}^{N} \left(x - a_i - b_{\sigma(i)}\right)$$

S_N: permutations of N elements

Theorem 4: If $\{a_i\}$ and $\{b_i\}$ are real, then so are $\{c_i\}$.

We will not prove this theorem in class. However, it was this preservation of real-rootedness, which motivated the first study of this operation ≈100 years ago.

Theorem 5: Let $P_k(x)$ and $\widetilde{P}_k(x)$ be Appell sequences corresponding to $\Phi(z)$ and $\widetilde{\Phi}(z)$ as in Theorem 1.

Then $[P_k \oplus \widetilde{P}_k](x)$ is the Appell sequence corresponding to $\Phi(z) \cdot \widetilde{\Phi}(z)$

We postpone the proof till midsemester homework

Example 3: If $P_k(x)$ corresponds to $\Phi(z)$, then $P_k(x-\gamma)$ corresponds to $e^{-\gamma z}\Phi(z)$

[Theorem 5 applied to $\widetilde{\Phi}(z) = e^{-\gamma z}$ using Example 5 and Theorem 3]

Exercise 3: Verify Theorem 5 on Exercises 1, 2 by showing that finite free convolution of polynomials in Exercise 1 with themselves gives polynomials in Exercise 2.