Math 833 Harmonic analysis Week 13

So far: classifications of extreme coherent systems
[what are all possible irreducibles?]

Today: decomposition of coherent systems into extremes
[Representation of a given system as a mixture.]

Analogies:

1. In Fourier analysis: decomposition of a given function as a combination of e^{icx}

2. In representation theory: decomposition of a given representation as a direct sum of irreducible representations

3. In random matrix theory: finding for a given invariant ensemble the law of eigenvalues (= decomposition into conjugation-orbits)

Example 1: Poisson-Dirichlet distribution

Recall partitition structures of Week 7, Slides 17-22

They are sequences of random set partitions of $\{1,..n\}$

$\pi_1 \leftarrow \pi_2 \leftarrow \pi_3 \leftarrow \pi_4 \leftarrow \dots$

With the law of π_n being $S(n)$-invariant and projection $\pi_n \to \pi_{n-1}$ being "forgetting" of n.

We discussed that extreme set partitions are parameterized by $\alpha_1 \geq \alpha_2 \geq .. \geq 0$; $\sum_i \alpha_i \leq 1$ [Asymptotic sizes of blocks divided by n]

Here is a not extreme partition structure: $\sigma: \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{matrix}$ | $(12)(345)$ $\{1,2\} \sqcup \{3,4,5\}$

Take π_n = cycles in uniformly random $\sigma \in S(n)$ [viewed as sets]

Clearly, this gives rise to a coherent system.

Definition: The Poisson-Dirichlet distribution is the law of infinite (random) sequence $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$, $\sum \alpha_i \leq 1$ governing the decomposition of π_n = cycles of uniform $S(n)$ into extreme systems

"Ergodic decomposition"

Theorem 1: The P-D distribution can be obtained through the stick-breaking process: for $u_i \sim$ i.i.d. uniform on $[0,1]$

$$(\alpha_1 \geq \alpha_2 \geq \dots) = \text{reordering}\big(u_1, (1-u_1)u_2, (1-u_1)(1-u_2)u_3, \dots\big)$$

Take a stick, break it at a random point

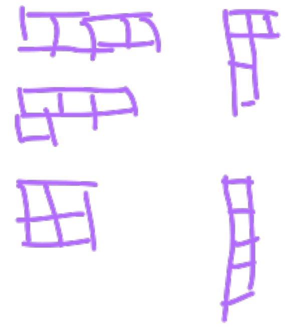
Keep the left one and break the right one again

Break the right one again, etc.

Proof of Theorem 1: Take uniformly random $\sigma \in S(n)$ and let $\{c_i\}$ be its lengths of cycles

Exercise 1: Take $n=4$. Find the law of $\{c_i\}$

[As a probability distribution on Young diagrams with 4 boxes]



Claim 1: Random $d_1 \geq d_2 \geq \dots$ governing decomposition has the the law of $\lim_{n\to\infty} \text{reordering} \left(\frac{c_i}{n}\right)$

[We do not give a proof, but this is because for extreme measures this is precisely the known law of large numbers]

Conclusion: The $n \to \infty$ limit of $\left(\frac{c_i}{n}\right)$ exists and it remains to identify it with the stick-breaking process

It is hard to work with unordered $\left(\frac{c_i}{n}\right) \Rightarrow$ let's introduce order

Given $\{c_i\}$ with $\sum_i c_i = n$, we define the vector $(r_1, r_2, \ldots)$ through a random procedure: [No requirement $r_1 \geq r_2 \geq \ldots$!]

- $r_1 = c_{i_1}$ with i_1 $\in \{1,2,\ldots\}$ chosen with probability $\frac{c_{i_1}}{n}$
  [$\frac{c_i}{n} \geq 0$, $\sum_i \frac{c_i}{n} = 1$. Thus, they define a probability distribution]
- Remove c_{i_1} from $\{c_i\}$ and repeat the same procedure for $r_2 = c_{i_2}$: i_2 is chosen with probability $\frac{c_{i_2}}{n - c_{i_1}}$
  [$\sum_{i \neq i_1} \frac{c_i}{n - c_{i_1}} = 1$]
- Etc

As a result, $(r_1, r_2, \ldots)$ is a random reordering of $\{c_i\}$

Claim 2: $\text{Prob}(r_1 = k) = \frac{1}{n}$, for all $k = 1, 2, \ldots, n$.

Proof of claim: The law of r_1 is the same as the law of the length of cycle in σ which contains 1
[Because probabilities $1 \in$ cycle c_i are precisely c_i/n]

Hence, $\mathrm{Prob}(r_1 = k) = \frac{1}{n!} \cdot \#\left\{\begin{matrix}\text{permutations with cycle of 1 having} \\ \text{length } k\end{matrix}\right\}$

$$= \frac{1}{n!} \underbrace{\binom{n-1}{k-1}}_{\text{choose elements in cycle}} \cdot \underbrace{(k-1)!}_{\text{order them}} \cdot (n-k)! \leftarrow \text{arbitrary permutation outside cycle}$$

$$= \frac{1}{n!} \frac{(n-1)!}{(k-1)!\,(n-k)!} \cdot (k-1)!\,(n-k)! = \frac{1}{n} \qquad \blacksquare$$

We now finish the proof of Theorem 1: the last claim implies that $\frac{r_1}{n} \to u_1 \sim \mathrm{Uniform}[0,1]$ as $n \to \infty$. Conditional on $\{r_1 = k\}$ the permutation outside the k-cycle is a uniformly random element of $S(n-k)$. Hence, conditional limit of r_2 given r_1 is computed by the same claim: $\frac{r_2}{n - r_1} \to u_2 \sim \mathrm{uniform}[0,1]$. Therefore,

$\frac{r_2}{n} = \left(1 - \frac{r_1}{n}\right)\left(\frac{r_2}{n-r_1}\right) \to (1-u_1) \cdot u_2$. Continuing further, we get the answer of Theorem 1. $\blacksquare$

Poisson-Dirichlet distribution is a remarkable object in probability. Some further reading:

terrytao.wordpress.com

The Poisson-Dirichlet process, and large prime factors of a random number

21 September, 2013 in expository, math.CO, math.NT | Tags: Dickman's function, permutations, point processes, Poisson-Dirichlet process, prime numbers

review by Tao

The Annals of Probability
1997, Vol. 25, No. 2, 855–900

THE TWO-PARAMETER POISSON–DIRICHLET DISTRIBUTION DERIVED FROM A STABLE SUBORDINATOR[1]

By Jim Pitman and Marc Yor

University of California, Berkeley

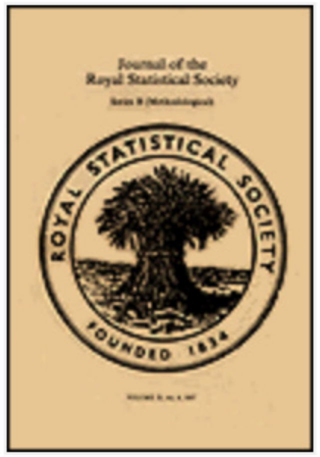
The two-parameter Poisson–Dirichlet distribution, denoted PD(α, θ), is a probability distribution on the set of decreasing positive sequences with sum 1. The usual Poisson–Dirichlet distribution with a single parameter θ, introduced by Kingman, is PD($0, \theta$). Known properties of PD($0, \theta$), including the Markov chain description due to Vershik, Shmidt and Ignatov, are generalized to the two-parameter case. The size-biased random permutation of PD(α, θ) is a simple residual allocation model proposed by Engen in the context of species diversity, and rediscovered by Perman and the authors in the study of excursions of Brownian motion and Bessel processes. For $0 < \alpha < 1$, PD($\alpha, 0$) is the asymptotic distribution of ranked lengths of excursions of a Markov chain away from a state whose recurrence time distribution is in the domain of attraction of a stable law of index α. Formulae in this case trace back to work of Darling, Lamperti and Wendel in the 1950s and 1960s. The distribution of ranked lengths of excursions of a one-dimensional Brownian motion is PD($1/2, 0$), and the corresponding distribution for a Brownian bridge is PD($1/2, 1/2$). The PD($\alpha, 0$) and PD(α, α) distributions admit a similar interpretation in terms of the ranked lengths of excursions of a semistable Markov process whose zero set is the range of a stable subordinator of index α.

2-parametric generalization

JOURNAL ARTICLE

Random Discrete Distributions

J. F. C. Kingman



Journal of the Royal Statistical Society. Series B (Methodological)
Vol. 37, No. 1 (1975), pp. 1-22 (22 pages)

Published by: Wiley for the Royal Statistical Society

Early appearance

Example 2: Hua-Pickrell measures (Weeks 9 and 10)

We now deal with the graph of spectra $\mathbb{S}p$ at $\beta=2$, i.e. with infinite Hermitian matrices invariant under $\mathcal{U}(\infty)$ conjugations.

Natural examples of coherent systems are Wigner $\frac{X+X^*}{2}$ and Wishart XX^* matrices. But they are extreme, hence, do not lead to any meaningful harmonic analysis.

Another simple measure: uniform (Haar) measure on $\mathcal{U}(N)$. But this is not a Hermitian matrix! Is there a way to make a coherent system out of it?

This can be done using the Cayley transform

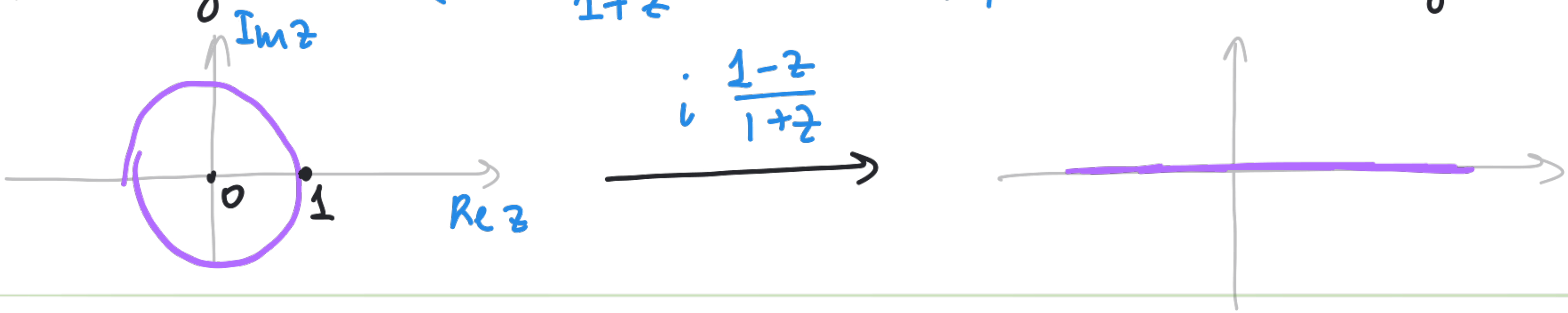
$$H \leftrightarrow U = \frac{i-H}{i+H} \quad \text{or} \quad U \leftrightarrow H = i\,\frac{1-U}{1+U}$$

Hermitian   unitary   Hermitian

$1 = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$

$i = \sqrt{-1} \cdot 1$

Explanation: let z and x be two complex numbers related by $x = i\,\frac{1-z}{1+z}$ then $|z|=1$ if and only if $x \in \mathbb{R}$



Exercise 2: Let H and U be two $N \times N$ matrices.

I) Show that $U = \frac{i-H}{i+H}$ is equivalent to $H = i\,\frac{1-U}{1+U}$

II) Show that if U and H are related by I), then U is unitary if and only if H is Hermitian

---

Remark 2: Cayley transform gives a good coordinate system on $\mathcal{U}(N)$.

Theorem 2: Let U_N be a uniformly random $N \times N$ unitary matrix and define $H_N := i\,\frac{1-U_N}{1+U_N}$. Then

- The law of H_N is invariant under unitary conjugations
- The law of $K \times K$ corner of H_N = law of H_K $(K < N)$

Hence, $(H_N)_{N \geq 1}$ define a $U(\infty)$-invariant infinite Hermitian matrix and the laws of eigenvalues of H_N form a coherent system on $\mathbb{S}_P$ $(\beta=2)$.

Sketch of the proof of Th. 2: First, the law of U_N is invariant under unitary conjugation, hence, so is the law of H_N. Next, take $K = N-1$. Then the identity in law

$$H_N \big|_{(N-1)\times(N-1)} \overset{d}{=} H_{N-1}$$

is equivalent to the following statement:
[By using Cayley transform twice]

Write $U_N = \begin{array}{c} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{array}{l} \updownarrow N-1 \\ \updownarrow 1 \end{array} \\ \begin{array}{cc} \overleftrightarrow{N-1} & \overleftrightarrow{1} \end{array} \end{array}$. Then $A - B(1+D)^{-1}C \overset{d}{=} U_{N-1}$

Projection from $U(N)$ to $U(N-1)$

In order to prove the last fact, one first shows that $A - B(1+D)^{-1} C$ is a $(N-1)\times(N-1)$ unitary matrix

[A good exercise in linear algebra!]

Then we notice that the probability distribution of this matrix is invariant under multiplications by $(N-1)\times(N-1)$ unitary matrices (because so is the law of $\mathcal{U}_N$).

Hence, by uniqueness of the Haar measure on unitary group $\mathcal{U}(N)$, this matrix has the same law as $\mathcal{U}_{N-1}$. $\boxtimes$

Remark 2: We are not going to prove it here, yet the law of H_N in Theorem 2 is explicit. It has density proportional to $\det(1+XX^*)^{-N}$ with respect to the Lebesgue measure on the $N\times N$ Hermitian matrices X.

At $N=1$ this is the Cauchy distribution of density $\sim \frac{1}{1+x^2}$

Theorem 3: Let (α_i), γ_1, γ_2 be random parameters, describing the decomposition of the coherent system in Theorem 2 into ergodic components. Then almost surely $\gamma_2 = 0$ and $\left\{\frac{1}{\pi \alpha_i}\right\}_{i=1}^{\infty}$ is the sine process of density 1, which is a transationally invariant point process on $\mathbb{R}$

such that its correlation functions (densities) ρ_k

$$\rho_k(x_1, \dots, x_k) := \lim_{\varepsilon \to 0} \varepsilon^{-k} \operatorname{Prob}\left(\text{there are particles in } \left[x_j - \tfrac{\varepsilon}{2}, x_j + \tfrac{\varepsilon}{2}\right],\ 1 \le j \le k\right)$$

are

$$\rho_k(x_1, \dots x_k) = \det \left[\frac{\sin(\pi(x_a - x_b))}{\pi (x_a - x_b)} \right]_{a,b=1}^{k}$$

if $a=b$, then this is 1

$\rho_1(x)$ = "density of points"

$\int_A \rho_1(x)\,dx$ = average # points in set A

The sine process is the universal scaling limit for the eigenvalues of random complex Hermitian matrices in the bulk of their spectra.

Infinite Random Matrices and Ergodic Measures

Alexei Borodin & Grigori Olshanski

Communications in Mathematical Physics **223**, 87–123(2001) | Cite this article

Problem setup + sine process

ELSEVIER
Advances in Mathematics
Volume 308, 21 February 2017, Pages 1209-1268

Infinite random matrices & ergodic decomposition of finite and infinite Hua–Pickrell measures

Yanqi Qiu [1]

description of γ_1

Sketch of the proof of Theorem 3. [Many steps are missing]

Let $(\lambda_1, \lambda_2, \dots, \lambda_N)$ be eigenvalues of H_N. Then the point process $\{\alpha_j\}_{j=1}^{\infty}$ is the $N\to\infty$ limit of $\{\frac{\lambda_i}{N}\}_{i=1}^{N}$.

[Which we consider only outside 0, ignoring the converging to 0 part]

For ergodic measures we proved this, and, with some technicalities, the result for general measures follows.

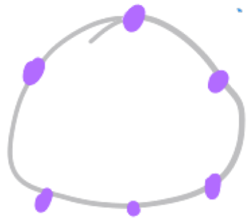
By the Cayley transform, with $v_1, \dots v_N$ - eigenvalues of U_N

$$v_j + 1 = \frac{i-\lambda_j}{i+\lambda_j} + 1 = \frac{2i}{i+\lambda_j} = \frac{1}{N}\,\frac{2i}{\frac{i}{N}+\frac{\lambda_j}{N}}.$$

Hence, for large N

$$\frac{N}{2i}(v_j+1) \approx \left(\frac{\lambda_j}{N}\right)^{-1} \Rightarrow \left\{\frac{1}{\pi \alpha_j}\right\} \approx \frac{N}{2\pi i}(v_j+1)$$

v_j are N points on the unit circle



The length of the circle is 2π, hence, average density of these points $\rho_1 = \frac{N}{2\pi}$. Thus, $\left\{\frac{N}{2\pi i}(v_j+1)\right\}$ as $N \to \infty$ becomes a point process on $\mathbb{R}$ of density $\rho_1(x) = 1$.

In order to see that this is, indeed, the sine process, one can use the following auxiliary claim [We do not give its proof.]

Claim 3: Correlation functions of $(v_1, \dots, v_N)$ are explicit:

$$\rho_k(e^{i\theta_1}, \dots, e^{i\theta_k}) = \det\left[\frac{1}{2\pi} \frac{\sin\left(\frac{N}{2}(\theta_a - \theta_b)\right)}{\sin\left(\frac{1}{2}(\theta_a - \theta_b)\right)}\right]_{a,b=1}^{k}$$

where $0 \le \theta_j < 2\pi$ are coordinates on the circle.

It remains to take $N \to \infty$ limit with $\theta_j = \pi + \frac{2\pi}{N} x_j$. [$e^{i\theta_j} \approx -1$]

Further reading:

arXiv.org > math > arXiv:math/9810015

Search...

Help | Ac

Mathematics > Representation Theory

[Submitted on 3 Oct 1998]

Point processes and the infinite symmetric group. Part VI: Summary of results

Alexei Borodin, Grigori Olshanski

Harmonic analysis on the infinite symmetric group

Sergei Kerov, Grigori Olshanski & Anatoly Vershik

Inventiones mathematicae **158**, 551–642(2004) | Cite this article

← Young graph

Annals of Mathematics, **161** (2005), 1319–1422

Harmonic analysis on the infinite-dimensional unitary group and determinantal point processes

By Alexei Borodin and Grigori Olshanski

← Gelfand-Tsetlin graph →

ELSEVIER

Journal of Functional Analysis

Volume 205, Issue 2, 20 December 2003, Pages 464-524

The problem of harmonic analysis on the infinite-dimensional unitary group

Grigori Olshanski

Journal of Functional Analysis

Volume 270, Issue 1, 1 January 2016, Pages 375-418

← q-GT

A quantization of the harmonic analysis on the infinite-dimensional unitary group

Vadim Gorin [a, b] ... Grigori Olshanski [b, c]

≈ GT for other root systems →

ELSEVIER

Advances in Mathematics

Volume 334, 20 August 2018, Pages 1-80

BC type z-measures and determinantal point processes

Cesar Cuenca

arXiv.org > math > arXiv:2009.04762

Search...

Help | Advar

Mathematics > Probability

[Submitted on 10 Sep 2020]

Infinite p-adic random matrices and ergodic decomposition of p-adic Hua measures

Theodoros Assiotis

p-adic version

general-β GT

Probability Measures on Dual Objects to Compact Symmetric Spaces and Hypergeometric Identities

G. I. Olshanski

Functional Analysis and Its Applications **37**, 281–301(2003) | Cite this article

We end this class with some open problems.

I) Prove the LLN for extreme coherent systems of Macdonald-deformed Young graph (Week 7, Slide 15)

Kerov's conjecture / Matveev's theorem say that they are parameterized by (α_i, β_i). Need to show $\frac{\lambda_i}{n} \to \alpha_i$, $\frac{\lambda_i'}{n} \to \beta_i$

II) Prove that structure coefficients of multiplication of multivariate Bessel functions are non-negative (Week 11, Slide 23):

$$B_{(x_1,\dots,x_N)}(z_1,\dots,z_N;\theta)\, B_{(y_1,\dots,y_N)}(z_1,\dots,z_N;\theta) = \int_{(v_1,\dots,v_N)} \underbrace{C^{v_1,\dots,v_N}_{x_1,\dots,x_N;\,y_1,\dots,y_N}}_{\geq 0\ ?}\, B_{(v_1,\dots,v_N)}(z_1,\dots,z_N;\theta)$$

More generally, for Macdonald polynomials:

$$P_\lambda(z_1,\dots,z_N;q,t)\cdot P_\mu(z_1,\dots,z_N;q,t) = \sum_\nu \underbrace{C^{\nu}_{\lambda,\mu}(q,t)}_{\geq 0 \text{ at } 0<q<1,\ 0<t<1\ ?}\, P_\nu(z_1,\dots,z_N;q,t)$$

Versions

JOURNAL ARTICLE
A Positive Radial Product Formula for the Dunkl Kernel
Margit Rösler

Transactions of the American Mathematical Society
Vol. 355, No. 6 (Jun., 2003), pp. 2413-2438 (26 pages)
Published by: American Mathematical Society

Advances in Mathematics
Volume 77, Issue 1, September 1989, Pages 76-115

Some combinatorial properties of Jack symmetric functions
Richard P Stanley

Conjecture 8.3

Crystallization of Random Matrix Orbits
Vadim Gorin, Adam W Marcus
International Mathematics Research Notices, Volume 2020, Issue 3, February 2020, Pages 883–913,
https://doi.org/10.1093/imrn/rny052
Published: 03 April 2018 Article history

Conjecture 2.1

Macdonald-positive specializations of the algebra of symmetric functions: Proof of the Kerov conjecture
Konstantin Matveev
Annals of Mathematics
Vol. 189, No. 1 (January 2019), pp. 277-316 (40 pages)

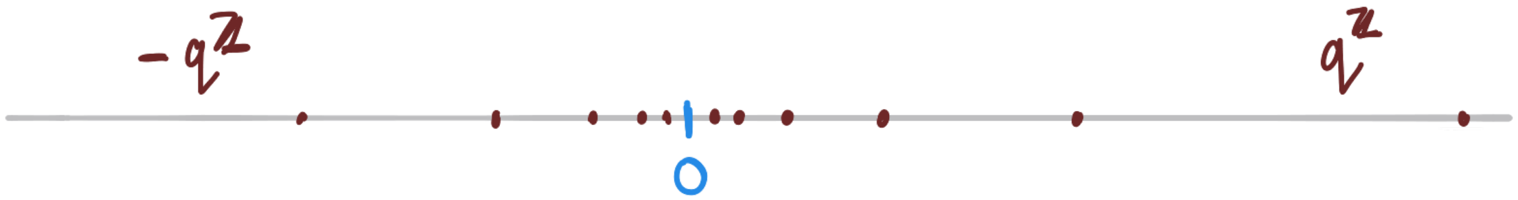
Section 1.2

III) What is the representation-theoretic meaning of the double q-lattice $\{q^{\mathbb{Z}}\} \cup \{-q^{\mathbb{Z}}\}$ appearing in the harmonic analysis on q-Gelfand-Tsetlin graph? (Week 12, Slide 23)

Journal of Functional Analysis
Volume 270, Issue 1, 1 January 2016, Pages 375-418

A quantization of the harmonic analysis on the infinite-dimensional unitary group

Vadim Gorin [a, b] ... Grigori Olshanski [b, c]



IV) Is there stochastic monotonicity for Macdonald-deformed Young and Gelfand-Tsetlin graphs? (Week 7, Slide 9)

For ordinary Young and GT graphs it is known to be true. For instance, for Young in says that if $|\mu|=|\lambda|=n$ and $\mu \geq \lambda$ in dominance order, then one can couple measures $L_{n\to n-1}(\lambda,\cdot)$ and $L_{n\to n-1}(\mu,\cdot)$, so that the inequality continues to hold.

Stochastic Monotonicity in Young Graph and Thoma Theorem

Alexey Bufetov, Vadim Gorin

International Mathematics Research Notices, Volume 2015, Issue 23, 2015, Pages 12920–12940,
https://doi.org/10.1093/imrn/rnv085

← see conjectures 1.6 and 2.4 for the Hall-Littlewood version.

The above four conjectures are challenging.

There are also many entry-level research problems which I am glad to share with anyone interested. Feel free to reach out to me for that!

The class on the boundaries has reached its boundary!