Math 833 General β random matrices Week 11

Over the last two lectures we dealt with complex Hermitian random matrices.

- Why Hermitian? Important for applications (quantum mechanics [where GUE first appeared], statistics [Wishart = sample covariance for data in frequencies space after Fourier], connections to statistical mechanics models [tilings, particle systems]). Guarantees that spectrum is real.

- Why complex? While we arrived at the complex setting naturally through the Gelfand-Tsetlin graph, real symmetric random matrices are equally (if not more) important.

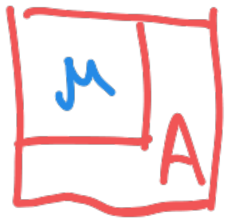
In addition, there are quaternionic Hermitian matrices.

Theorem 1: Let A be a $N \times N$ random Hermitian matrix with eigenvalues $\lambda_1 \geq \dots \geq \lambda_N$ and uniformly random eigenvectors. Can be:

- Real $(\beta = 1)$: invariant under $A \to OAO^*$ for orthogonal O
- Complex $(\beta = 2)$: invariant under $A \to uAu^*$ for unitary u
- Quaternionic $(\beta = 4)$; invariant under $A \to sAs^*$ for symplectic s

[In each case invariance under the group preserving $\langle x, y \rangle = \sum_{i=1}^{N} x_i \bar{y}_i$]

Let $\mu_1 \geq \dots \geq \mu_{N-1}$ denote the eigenvalues of $(N-1) \times (N-1)$ submatrix of A



Then the law of (μ_i) given (λ_i) is the same as $(N-1)$ zeros of $\displaystyle\sum_{i=1}^{N} \frac{\xi_i}{\mu - \lambda_i} = 0$, where ξ_i are i.i.d.

$$\xi_i \overset{d}{\sim} \chi^2_\beta = \text{sum of } \beta \text{ independent } (N(0,1))^2$$

Remark: A direct computation shows that the density of χ^2_β is:

$$P_\beta(x) = \frac{1}{2^{\beta/2}\,\Gamma(\frac{\beta}{2})} x^{\beta/2 - 1} e^{-x/2}, \quad x > 0.$$

Proof of Theorem 1: The argument is the same as Week 10, Corollary 3: the eigenvalues (μ_i) solve

$$\sum_{a=1}^{N} \frac{z_a \overline{z_a}}{\mu - \lambda_a} = 0,$$

where z_a are i.i.d. and

$z_a \sim N(0,1)$ at $\beta = 1$, $z_a \sim N(0,1) + i N(0,1)$ at $\beta = 2$, and $z_a \sim N(0,1) + i N(0,1) + j N(0,1) + k N(0,1)$ at $\beta = 4$. $\square$

Remark: For $\beta = 4$ case one needs to be more careful, since quaternions do not commute and the notions of determinant and eigenvalue need clarifications. We will not detail this here.

Observation: The χ^2_β distributions in the answer actually make sense for any real value of $\beta > 0$!

Definition: The β-corners (or β-Spectra) branching graph has levels $Sp_N^\beta = \{x_1 \geq x_2 \geq \dots \geq x_N \mid x_i \in \mathbb{R}\}$ and cotransition probabilities $L_{N\to(N-1)}((x_1,\dots x_N) \to (y_1,\dots y_{N-1}))$

given by the law of $N-1$ zeros of the polynomial in y

$$(*) \qquad \prod_{i=1}^{N} (y - x_i) \cdot \sum_{i=1}^{N} \frac{\xi_i}{y - x_i}$$

, where $\xi_i \geq 0$ are i.i.d.

χ^2_β with density $p_\beta(x) = \frac{1}{2^{\beta/2}\Gamma(\frac{\beta}{2})} x^{\beta/2-1} e^{-x/2}$, $x>0$

$\beta>0$ can be arbitrary real in this definition, but by Theorem 1 at $\beta=1,2,4$ the graph arises as spectra of corners of Hermitian real/complex/quaternion matrices

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Since $(*)$ changes its sign on each interval $[x_i, x_{i+1}]$, we have almost sure interlacement: $x_1 \geq y_1 \geq x_2 \geq \dots \geq y_{N-1} \geq x_N$

Theorem 2: The link $L_{N\to N-1}$ is a measure of density

$$\frac{\Gamma(\frac{\beta}{2}N)}{\Gamma(\frac{\beta}{2})^N} \mathbb{1}_{x_1>y_1>\dots>y_{N-1}>x_N} \frac{\prod_{1\le i<j<N}(y_i-y_j)}{\prod_{1\le i<j\le N}(x_i-x_j)^{\beta-1}} \prod_{i=1}^{N-1}\prod_{j=1}^{N}|y_i-x_j|^{\frac{\beta}{2}-1} dy_1\dots dy_{N-1}$$

[If $x_1>\dots>x_N$ and extended by continuity to the case of coinciding x_i]

Further, for the array $(y_i^k)_{1\le i\le k}$ distributed according to a central measure on paths in $\mathbb{S}p^\beta$, the conditional law of $(y_i^k)_{1\le i\le k<N}$ given $(y_1^N,\dots y_N^N)=(x_1,\dots x_N)$ [i.e., fixed]

has density $\sim$ up to a normalization

$$\prod_{k=1}^{N-1}\left[\prod_{1\le i<j\le k}(y_i^k-y_j^k)^{2-\beta}\prod_{i=1}^{k}\prod_{j=1}^{k+1}|y_i^k-y_j^{k+1}|^{\frac{\beta}{2}-1}\right]$$

note different signs and vanishing at $\beta=2$

with respect to the Lebesgue measure on the polytope defined by the interlacing conditions $y_i^{k+1}\geq y_i^k\geq y_{i+1}^{k+1}$, $1\le i\le k<N$.

Proof of Theorem 2: Note that the conditional law of the array (y_i^k) is obtained by multiplying densities of the links $L_{k+1\to k}$ over $k=1,2,\dots,N-1$. Hence, we only deal with links.

We take $(*)$ and divide it by $\sum_{i=1}^{N} z_i$, getting the equation

$$\sum_{i=1}^{N} \frac{w_i}{y-x_i}=0, \quad w_i=\frac{z_i}{\sum_{j=1}^{N} z_j}: \quad w_i \geq 0, \ \sum_{i=1}^{N} w_i=1$$

Using the explicit density of χ^2_β distributions, the joint density of (z_i) is $\sim \prod_{i=1}^{N} (z_i)^{\beta/2-1} \cdot \exp\left(-\frac{z_1+\dots+z_N}{2}\right)$

When we condition on the value of $z_1+\dots+z_N$, exponent is constant.

Hence, the law of $(w_i)_{i=1}^{N}$ is the Dirichlet distribution $D(\frac{\beta}{2},\dots,\frac{\beta}{2})$:

$$\frac{\Gamma\left(\frac{\beta}{2}N\right)}{\Gamma\left(\frac{\beta}{2}\right)^N} \prod_{i=1}^{N} (w_i)^{\beta/2-1} \, dw_1 \cdots dw_{N-1}, \quad w_i \geq 0, \ \sum_{i=1}^{N} w_i = 1.$$

Normalization constant making total mass = 1.

In the proof of Week 10, Proposition 2, we found that

$$w_i = \frac{\prod_{j=1}^{N-1}(x_i - y_j)}{\prod_{j\neq i}(x_i - x_j)} \quad \text{and} \quad dw_1\cdots dw_{N-1} = \frac{\prod_{i<j}(y_i-y_j)}{\prod_{i<j}(x_i-x_j)}\, dy_1\cdots dy_{N-1}$$

Thus, $$\frac{\Gamma(\frac{\beta}{2}N)}{\Gamma(\frac{\beta}{2})^N}\prod_{i=1}^{N}(w_i)^{\beta/2-1}\, dw_1\cdots dw_{N-1} =$$

$$= \frac{\Gamma(\frac{\beta}{2}N)}{\Gamma(\frac{\beta}{2})^N}\cdot\prod_{i=1}^{N}\prod_{j=1}^{N-1}|x_i-y_j|^{\beta/2-1}\cdot\prod_{i=1}^{N}\prod_{j\neq i}\frac{1}{|x_i-x_j|^{\beta/2-1}}\cdot\frac{\prod_{i<j}(y_i-y_j)}{\prod_{i<j}(x_i-x_j)}\, dy_1\cdots dy_{N-1}$$

The transition densities $L_{N\to N-1}$ and their normalization is old:

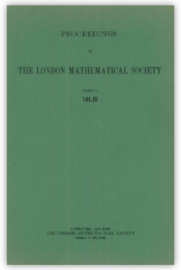
Proceedings of the London Mathematical Society

LONDON MATHEMATICAL SOCIETY EST. 1865

Papers

Generalization of Legendre's Formula
$KE' - (K-E)K' = \frac{1}{2}\pi$

A. L. Dixon



Volume s2-3, Issue 1
1905
Pages 206-224

Well before the birth of the Random matrix theory

A Short Proof of Selberg's Generalized Beta Formula.

Greg W. Anderson

Forum mathematicum (1991)
Volume: 3, Issue: 4, page 415-418

$$\int_0^1\cdots\int_0^1 \prod_{i<j}|x_i-x_j|^{\beta}\prod_{i=1}^{N}(x_i)^{\alpha-1}(1-x_i)^{\gamma-1}dx_i = ?$$

Exercise 1: Taking a limit from Theorem 2, find

$$L_{4\to 3}\big((1,1,0,0)\to(y_1,y_2,y_3)\big)$$

Hint: Consider $L_{4\to3}\big((1+\varepsilon,1,\varepsilon,0)\to(y_1,y_2,y_3)\big)$ and send $\varepsilon\to 0$.

Remark: The total weight $\int_{(y_i)} L_{N\to N-1}\big((x_1,\ldots,x_N)\to(y_1,\ldots,y_{N-1})\big)$ is always 1, including the above exercise.

This leads to many integral evaluations, which are particular cases of quite general Dixon-Anderson integral, from two papers on the previous slide.

Proposition 1: $\lim_{\beta\to\infty} L_{N\to N-1}$ of Sp^β is the deterministic transition $(x_1,\dots,x_N)\to(y_1,\dots,y_{N-1})$: $\prod_{i=1}^{N-1}(z-y_i)=\frac{1}{N}\frac{\partial}{\partial z}\prod_{i=1}^{N}(z-y_i)$

Appell sequences of Week 2 are $\beta\to\infty$ limits of eigenvalues of corners of Hermitian matrices

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Proof of Proposition 1: (y_i) are the roots of

$$\frac{1}{N}\prod_{i=1}^{N}(y-x_i)\sum_{i=1}^{N}\frac{\xi_i/\beta}{y-x_i}=0 \qquad (**)$$

[we divided by βN]

Recall that ξ_i is a sum of β independent $N(0,1)^2$.
$\mathbb{E}\,\xi_i=\beta$ and $\mathrm{Var}(\xi_i)=2\beta$. [Also true for non-integral β]
Therefore, $\lim_{\beta\to\infty}\frac{\xi_i}{\beta}=1$, and $(**)$ turns into

$$\frac{1}{N}\prod_{i=1}^{N}(y-x_i)\sum_{i}\frac{1}{y-x_i}=\frac{1}{N}\frac{\partial}{\partial y}\prod_{i=1}^{N}(y-x_i).$$

$\blacksquare$

Conclusion: The links $L_{N\to N-1}$ in Sp^β are nice at $\beta = 1, 2, 4, \infty$, corresponding to real/complex/quaternion matrices and differentiation.

Question: Is our exterpolation from these four points to general $\beta > 0$ anyhow good or canonical?

Yes, because many structures get preserved:

- Connections to classical ensembles of random matrix theory, such as Wigner or Wishart matrices
- Connections to the theory of symmetric functions
- Agrees with Dyson Brownian Motion

Multilevel Dyson Brownian motions via Jack polynomials

Vadim Gorin & Mykhaylo Shkolnikov

Probability Theory and Related Fields **163**, 413–463(2015) | Cite this article

[N-dimensional diffusion, solving an SDE, generalizing the one for the evolution of spectrum of a Hermitian matrix of i.i.d. Brownian motions]

Theorem 3: Consider a measure G_N^β on Sp_N^β of density proportional to $\prod_{i<j}(x_i-x_j)^\beta \prod_{i=1}^N e^{-\frac{\beta}{4}(x_i)^2}$. Then:

I) The measures G_N^β form a coherent system on Sp_N^β

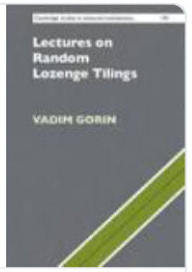
II) At $\beta=1,2,4$ the corresponding measure on infinite matrices is as follows: take an infinite matrix X with i.i.d. matrix elements: $N(0,2)$ at $\beta=1$, $N(0,1)+iN(0,1)$ at $\beta=2$, $N(0,\frac{1}{2})+iN(0,\frac{1}{2})+jN(0,\frac{1}{2})+kN(0,\frac{1}{2})$ at $\beta=4$

Set $M=\frac{1}{2}(X+X^*)$. Then G_N^β is the law of the eigenvalues of the principal $N\times N$ top-left submatrix of M

G_N^β is called **Gaussian β ensemble**. The corresponding central measure on paths in Sp^β is **Gaussian β corners process**.

For $\beta=1,2,4$, both parts can be proven simultaneously by using a very close argument to the proof of Week 10, Theorem 1

Lectures on Random Lozenge Tilings
Book by Vadim Gorin



See Section 20.1

http://www.math.wisc.edu/~vadicgor/Random_tilings.pdf

For general $\beta>0$ there are two appoaches:

Limit of discrete coherency relations proven through symmetric functions

Multilevel Dyson Brownian motions via Jack polynomials

Vadim Gorin & Mykhaylo Shkolnikov

Probability Theory and Related Fields **163**, 413–463(2015) | Cite this article

See Section 2.2.

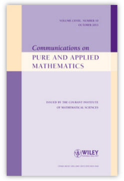
Dixon-Anderson integration identities [generalizing explicit formula for $L_{N\to N-1}$]

Communications on PURE AND APPLIED MATHEMATICS

Research Article

General β-Jacobi Corners Process and the Gaussian Free Field

Alexei Borodin, Vadim Gorin



Volume 68, Issue 10
October 2015
Pages 1774-1844

See Proposition 2.7 and take Gaussian limit

Similar statements hold for sample covariance matrices XX^*, corresponding to Laguerre β ensemble. We will not detail further.

In the previous two lectures we described the boundary of Sp^2 in terms of matrices or Fourier transforms. The first does not exist at general $\beta>0$. The second needs an adjustment.

Definition: Multivariate Bessel function $B_{(x_1,\dots x_N)}(z_1,\dots z_N;\beta)$ is a function of $x_1 \geq \dots \geq x_N$ and $z_i \in \mathbb{C}$ defined by

$$B_{(x_1,\dots x_N)}(z_1,\dots,z_N;\beta) = \mathbb{E}\, e^{\sum_{k=1}^{N} z_k \left(\sum_{i=1}^{k} y_i^k - \sum_{i=1}^{k-1} y_i^{k-1}\right)},$$

where $(y_i^k)_{1\le i\le k\le N}$ is distributed as a path is Sp^β ending at $(y_1^N,\dots y_N^N)=(x_1,\dots x_N)$, i.e. by the law of Slide 5.

Some properties:

The literature often uses $\theta=\frac{\beta}{2}$ as a parameter.

- $B_{(x_1,\dots x_N)}(z_1,\dots z_N)$ is symmetric in $z_1, z_2, \dots, z_N$
- $B_{(x_1,\dots,x_N)}(z_1,\dots z_N)$ is analytic (holomorphic) both in (z_i) and in (x_i)
- $B_{(x_1,\dots x_N)}(z_1,\dots z_N) = B_{(z_1,\dots z_N)}(x_1,\dots x_N)$; $B_{(x_1,\dots x_N)}(0,\dots 0) = 1$

We prove these properties <u>only</u> at $\beta=2$ [complex Hermitian case]

A central tool is the computation of the orbital integral or Fourier transform of the uniform measure on fixed spectra matrices.

Theorem 4: Take $x_1 \geq \dots \geq x_N$ and uniformly random $N \times N$ Hermitian matrix A with such eigenvalues. Let Z be another $N \times N$ matrix [diagonalizable by unitary conjugation] with eigenvalues $\{z_i\}$.

Then
$$\mathbb{E}_A\, e^{\operatorname{Trace}(ZA)} = \int_{U(N)} e^{\operatorname{Trace}\left(Z u \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix} u^*\right)} du$$
uniform (Haar) measure on $U(N)$

$$= \prod_{k=1}^{N-1} k! \cdot \frac{\det\left[e^{z_i x_j}\right]_{i,j=1}^{N}}{\prod_{i<j}(x_i - x_j)(z_i - z_j)} = B_{(x_1,\dots x_N)}(z_1,\dots z_N;\ \beta=2)$$

(D)

Harish-Chandra, Itzykson, Zuber } integral

The planar approximation. II

Journal of Mathematical Physics 21, 411 (1980); https://doi.org/10.1063/1.524438

C. Itzykson and J.-B. Zuber

Differential Operators on a Semisimple Lie Algebra

Harish-Chandra

American Journal of Mathematics
Vol. 79, No. 1 (Jan., 1957), pp. 87-120 (34 pages)

Proof of Theorem 4: If we replace Z by wZw^*, where $w \in \mathcal{U}(N)$, then $\mathrm{Trace}(wZw^*A) = \mathrm{Trace}(Zw^*Aw)$, which has the same distribution as $\mathrm{Trace}(ZA)$, since A is invariant under conjugations.

Hence, $\mathbb{E}_A e^{\mathrm{Trace}(ZA)} = \mathbb{E}_A e^{\mathrm{Trace}\left(\begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_N \end{pmatrix} A\right)} = \mathbb{E}_A e^{z_1 a_{11} + \dots + z_N a_{NN}}$,

where $A = [a_{ij}]_{i,j=1}^N$.

Next, let A_K be $K \times K$ submatrix of A and let $y_1^K \geq y_2^K \geq \dots \geq y_K^K$ be its eigenvalues. We can write:

$$\sum_{K=1}^N z_K a_{KK} = \sum_{K=1}^N z_K \left(\mathrm{Trace}(A_K) - \mathrm{Trace}(A_{K-1})\right) = \sum_{K=1}^N z_K \left(\sum_{i=1}^K y_i^K - \sum_{i=1}^{K-1} y_i^{K-1} \right)$$

Comparing with definition of $B_{(x_1, \dots x_N)}(z_1, \dots z_N; 2)$, we get

$$\mathbb{E}_A e^{\mathrm{Trace}(ZA)} = B_{(x_1, \dots x_N)}(z_1, \dots, z_N; 2)$$

For the determinantal formula we use induction in N.

The definition of B yields the following recurrence:

$$B_{(x_1,\dots,x_N)}(z_1,\dots,z_N) = \int_{\substack{(y_1,\dots,y_{N-1}) \\ x_1 \geq y_1 \geq \dots \geq y_{N-1} \geq x_N}} \underbrace{(N-1)!\, \frac{\prod_{i<j}(y_i-y_j)}{\prod_{i<j}(x_i-x_j)}}_{\text{law of } (y_i) \text{ given } (x_i)} e^{z_N(\Sigma x_i - \Sigma y_i)} \cdot B_{(y_1,\dots,y_{N-1})}(z_1,\dots,z_{N-1})\, dy_1 \cdot \ldots \cdot dy_{N-1}$$

We need to check that the determinantal formula (D) also satisfies the same recurrence. This reduces to

$$\det\left[e^{x_i z_j}\right]_{i,j=1}^{N} \overset{?}{=} \prod_{i=1}^{N-1}(z_i - z_N) \int_{\substack{(y_1,\dots,y_{N-1}) \\ x_1 \geq y_1 \geq \dots \geq y_{N-1} \geq x_N}} e^{z_N(\Sigma x_i - \Sigma y_i)} \det\left[e^{y_i z_j}\right]_{i,j=1}^{N-1} dy_1 \cdot \ldots \cdot dy_{N-1}$$

In the RHS we can integrate explicitly in each $y_i \in [x_{i+1}, x_i]$ and get $e^{z_N \Sigma x_i} \det\left[\frac{e^{x_i(z_j - z_N)} - e^{x_{i+1}(z_j - z_N)}}{z_j - z_N}\right]_{i,j=1}^{N-1}$

Hence, it remains to check

$$\det\left[e^{x_i z_j}\right]_{i,j=1}^{N} \overset{?}{=} e^{z_N \sum_i x_i} \det\left[e^{x_i(z_j - z_N)} - e^{x_{i+1}(z_j - z_N)}\right]_{i,j=1}^{N-1} \quad (***)$$

The matrix in LHS is $\begin{bmatrix} \ddots & & e^{x_1 z_N} \\ & \ddots & \vdots \\ & & e^{x_N z_N} \end{bmatrix}$. Let us transofm it, preserving the determinant

We multiply 2nd row by $e^{(x_1 - x_2) z_N}$ and subract from the 1st one. Then we multiply 3rd row by $e^{(x_2 - x_3) z_N}$ and subtract from the 2nd one. And so on up to the N-th row. As a result, the last column becomes $\begin{bmatrix} 0 \\ \vdots \\ e^{x_N z_N} \end{bmatrix}$ and top-left submatrix ($(N-1)\times(N-1)$) becomes $\left[e^{x_i z_j} - e^{x_{i+1} z_j + (x_i - x_{i+1}) z_N}\right]$

Factoring out $e^{x_i z_N}$ from the i-th row, $i = 1, 2, \ldots, N$, we get the determinant in the RHS of $(***)$.

At $\beta=2$ the properties of $B_{(x_1,\dots x_N)}(z_1,\dots z_N)$ of Slide 13 are immediate from (Ⓓ) form of Theorem 3.

At general values of $\beta>0$, no simple explicit formulas for $B_{(x_1,\dots x_N)}(z_1,\dots z_N;\beta)$ exist. The ways people think about them:

- As eigenfunction of (explicit) difference/differential operator
  [At $N=1$, $e^{x_1 z_1}$ is an eigenfunction of $\frac{\partial}{\partial z_1}$ and of $z_1 \to z_1 + c$]
- As limits of Jack or Macdonald symmetric polynomials
  [At $N=1$, $e^{x_1 z_1} = \lim_{M\to\infty} \left(1+\frac{z_1}{M}\right)^{\lfloor M x_1 \rfloor}$]
- Through Taylor series expansion
  [At $N=1$, $e^{x_1 z_1} = \sum_{m=0}^{\infty} \frac{(x_1 z_1)^m}{m!}$]
- Through contour integral representations
  [As in Week 8, Slide 9 for the Schur functions].

Bessel functions give us a way to describe the boundary of Sp^β

Theorem 5: Extreme coherent systems on Sp^β are parameterized by (α_i), $\gamma_1 \in \mathbb{R}$, $\gamma_2 \geq 0$ satisfying $\sum_i (\alpha_i)^2 < \infty$. On $Sp_1^\beta = \mathbb{R}$ the law of the corresponding random variable η is given by

$\sqrt{-1}$

$$\mathbb{E}\, e^{iz\eta} = \Phi_\beta^{(\alpha,\gamma)}(iz) = e^{i\gamma_1 z} \cdot e^{-\gamma_2 \frac{z^2}{\beta}} \prod_{j=1}^{\infty} \frac{e^{-i\alpha_j z}}{\left(1 - i\frac{2}{\beta}\alpha_j z\right)^{\beta/2}}$$

More generally, if $(\eta_1 \geq \dots \geq \eta_N)$ is the corresponding r.v. on Sp_N^β, then

$$\mathbb{E}\, B_{(\eta_1,\dots,\eta_N)}(z_1,\dots,z_N) = \Phi_\beta^{(\alpha,\gamma)}(z_1) \cdot \Phi_\beta^{(\alpha,\gamma)}(z_2) \cdot \ldots \cdot \Phi_\beta^{(\alpha,\gamma)}(z_N)$$

random?

depend on N

Further, $\lim\limits_{N\to\infty} \frac{\eta_j}{N} = \alpha_j^+$, $\lim\limits_{N\to\infty} \frac{\eta_{N+1-j}}{N} = \alpha_j^-$, $\{\alpha\} = \{\alpha_j^+\} \cup \{\alpha_j^-\}$,

$$\lim_{N\to\infty} \frac{\eta_1 + \dots + \eta_N}{N} = \gamma_1, \quad \lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{\eta_i}{N}\right)^2 = \sum_{j=1}^{\infty} (\alpha_j)^2 + \gamma_2.$$

Sketch of proof. As in $\beta=2$, the proof splits into two parts: figuring out the law of $\eta \in Sp^{\beta}_1$ and showing multiplicativity. We only deal with the first part, which is based on an interesting auxiliary statement:

Lemma. Let $(y_i^k)_{1\le i\le k\le N}$ be distributed as a path in Sp^{β} ending at $(y_1^N, \dots y_N^N) = (x_1, \dots x_N)$, i.e. by the law of Slide 5.

Then $y_1^1 \overset{d}{=} w_1 x_1 + w_2 x_2 + \dots + w_N x_N$, where $(w_1, \dots, w_N)$ is Dirichlet-distributed $D(\frac{\beta}{2}, \dots, \frac{\beta}{2})$ (as on Slide 6)

Proof of Lemma: In $\beta=2$ case we used random matrix realization to prove this (Week 10, Slide 15), but there are no matrices at general $\beta>0$. Instead we rely on the [which we have not proven, sorry :)] symmetry of $B_{(x_1, \dots x_N)}(z_1, \dots z_N; \beta)$

The symmetry under $z_1 \leftrightarrow z_N$ means a distributional identity

$$y_1^1 \overset{d}{=} \sum_{i=1}^{N} x_i - \sum_{i=1}^{N-1} y_i^{N-1}$$

By definition of Sp^β, $\sum_{i=1}^{N-1} y_i^{N-1}$ is the sum of the zeros of

$$\prod_{i=1}^{N}(y-x_i)\cdot\sum_{i=1}^{N}\frac{w_i}{y-x_i} = \sum_{i=1}^{N} w_i \prod_{j\neq i}(y-x_j)$$

By the Vietta's formulas, this is minus the coefficient of y^{N-2}.

We get $\sum_{i=1}^{N} w_i \sum_{j\neq i} x_j = \underbrace{\sum_{i=1}^{N} w_i}_{=1} \cdot \sum_{j=1}^{N} x_j - \sum_{i=1}^{N} w_i x_i$. $\blacksquare$

Returning to the proof of the theorem, using the embedding minimal boundary $\subset$ Martin boundary, the law of $\eta\in Sp_1^\beta$ is the $N\to\infty$ limit in distribution of sums $\sum_{i=1}^{N} w_i x_i$, where $(w_1,\dots,w_N)\sim D(\frac{\beta}{2},\dots,\frac{\beta}{2})$ and $(x_1\geq\dots\geq x_N)$ is allowed to depend on N in an arbitrary deterministic way. Representing $w_i = \frac{\xi_i}{\sum_{j=1}^{N}\xi_j}$ with i.i.d. $\xi_i\sim\chi^2_\beta$, we are looking for the limits of

$$\frac{\beta N}{\sum_{j=1}^{N}\xi_j}\cdot\sum_{i=1}^{N}\frac{\xi_i}{\beta}\cdot\frac{x_i}{N}.$$

The prefactor $\to 1$ and can be ignored

For the sum $\sum_{i=1}^{N}$ we can compute the characteristic function

$$\prod_{i=1}^{N} \frac{1}{\left(1 - i \cdot \frac{2}{\beta} z \cdot \frac{x_i}{N}\right)^{\beta/2}} \qquad (\star)$$

because of independence of ξ_i

variable in char. function

characteristic function of χ^2_β (see Wiki)

We need to find all possible limits of $(\star)$, but up to rising into a power and rescaling z. this is the same task which we solved in Week 10 for $\beta=2$. (And in Week 2 for $\beta=\infty$)

Similarly to $\beta=2$ and $\beta=\infty$ cases, the function $\Phi_\beta^{(\alpha,\delta)}(z)$ in Theorem 4 is multiplicative. I.e., the general case is obtained from 3 basic examples: $\gamma_1 = 1$; $\gamma_2 = 1$; $\alpha_1 = 1$.

At $\beta=2$ we were adding independent matrices for that.

At $\beta=\infty$ we used finite free convolution. What about general $\beta>0$?

Definition: Given $(x_1,\dots x_N)\in Sp_N^\beta$, $(y_1,\dots y_N)\in Sp_N^\beta$, we define random $(v_1,\dots v_N) = (x_1,\dots,x_N) \boxplus_\beta (y_1,\dots y_N) \in Sp_N^\beta$ by requiring (Δ)

$$\forall z_1,\dots,z_N\in\mathbb{C}: \quad \mathbb{E}\, B_{(v_1,\dots v_N)}(z_1,\dots z_N) = B_{(x_1,\dots x_N)}(z_1,\dots z_N)\cdot B_{(y_1,\dots y_N)}(z_1,\dots z_N)$$

Tricky point: At $\beta=1,2,4,\infty$, this is equivalent to addition of matrices and finite free convolution. At general $\beta>0$, one needs to check that (Δ) defines a probality measure on Sp_N^β (law of $(v_1,\dots v_N)$). Total mass $=1$ is a simple exercise, but positivity is an open problem.

Exercise 2: Describe explicitly $\boxplus_\beta$ for $N=1$. How does it depend on β?

Starting from independent 3 basic examples and using $\boxplus_\beta$, we get arbitrary $(\alpha_i;\gamma_1,\gamma_2)$.

Further reading:

arXiv.org > math > arXiv:1905.08684

Mathematics > Probability

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The boundary of the orbital beta process

Theodoros Assiotis, Joseph Najnudel

Theorem 4

Crystallization of Random Matrix Orbits FREE

Vadim Gorin, Adam W Marcus

International Mathematics Research Notices, Volume 2020, Issue 3, February 2020, Pages 883–913,

$\boxplus_\beta$; $\beta\to\infty$; limit Jack $\to$ Bessel

Limit Jack $\to$ Bessel

Mathematical Research Letters 4, 69–78 (1997)

SHIFTED JACK POLYNOMIALS, BINOMIAL FORMULA, AND APPLICATIONS

A. Okounkov and G. Olshanski

Discrete version of Th. 4

Asymptotics of Jack polynomials as the number of variables goes to infinity

Andrei Okounkov, Grigori Olshanski

International Mathematics Research Notices, Volume 1998, Issue 13, 1998, Pages 641–682,