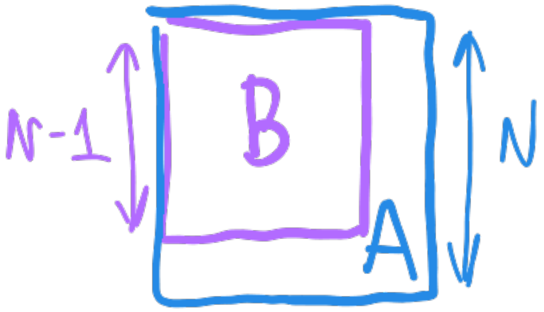
Math 833 — Spectra of Hermitian Matrices — Week 10

Our first task for today is to understand the interplay between cutting corners in Hermitian matrices and branching rules in the graph of spectra.

A general question that we are going to address:
Given $N \times N$ matrix A and $(N-1)\times(N-1)$ submatrix B, how are their eigenvalues related?



There are two ways to think about it:
[eventually, they are equivalent!]

- Start with B and grow A

See 20.1 in "Lectures on random lozenge tilings"
http://www.math.wisc.edu/~vadicgor/Random_tilings.pdf

- Start with A and restrict to B

Our approach for today

Proposition 1: Take a $N\times N$ Hermitian matrix [=quadratic form] $A=[a_{ij}]$ and a hyperplane $L=\{(x_i) \mid l_1x_1+\ldots+l_Nx_N=0\}$. Then $N-1$ eigenvalues μ_i of the restriction $A|_L$ of A to L are roots of degree $N-1$ polynomial equation in variable μ

$$\det\begin{bmatrix} a_{11}-\mu & a_{12} & \cdots & a_{1N} & l_1 \\ a_{21} & a_{22}-\mu & & a_{2N} & \vdots \\ \vdots & & \ddots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN}-\mu & l_N \\ \bar{l}_1 & \bar{l}_2 & \cdots & \bar{l}_N & 0 \end{bmatrix} = 0 \qquad (*)$$

(This is $(N+1)\times(N+1)$ matrix)

Proof: First, let us clarify what we mean by restriction. Let P denote the orthogonal projector onto L. Then, PAP can be treated as

- Quadratic form in L
- Self-adjoint operator in L
- Quadratic form in original space, vanishing on $L^\perp$
- Self-adjoint operator in the original space, preserving L and $L^\perp$; acting as 0 on $L^\perp$ [orthogonal complement to L]

In all these interpretations PAP has the same eigenvalues, with the only difference being an additional 0 eigenvalue in the last two interpretations (which we ignore).

Now take a unitary matrix $u \in U(N)$ and treat it as $(N+1)\times(N+1)$ matrix $\begin{pmatrix} u & \begin{smallmatrix}0\\ \vdots\\ 0\end{smallmatrix} \\ 0\cdots 0 & 1 \end{pmatrix}$. Multiply $(*)$ by u from the left and by u^* from the right.

This results in transformation $A \to u A u^*$, $\begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix} \to u \begin{pmatrix} \ell_1 \\ \vdots \\ \ell_N \end{pmatrix}$

and $PAP \to (uPu^*)\,u A u^*\,(uPu^*) = u\,PAP\,u^*$, which has the same eigenvalues as PAP.

In other words, this is a change of basis in the space, which does not change our problem. Hence, by using appropriate u, we can assume without loss of generality that $\ell_2 = \ell_3 = \dots = \ell_N = 0$, i.e. we are projecting on the last $(N-1)$ coordinate vectors. This case is obvious. ∎

Corollary 1: Suppose that A is diagonal: $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$, then eigenvalues of its restriction to $L = \{(x_i) \mid \sum l_i x_i = 0\}$ solve

$$\sum_{i=1}^{N} \frac{|l_i|^2}{\mu - \lambda_i} = 0,$$

which is a degree $(N-1)$ polynomial equation on μ after clearing the denominators.

Proof:

$$\det \begin{pmatrix} \lambda_1 - \mu & & 0 & l_1 \\ & \ddots & & \vdots \\ 0 & & \lambda_N - \mu & l_N \\ \bar{l}_1 & \dots & \bar{l}_N & 0 \end{pmatrix} = \sum_{i=1}^{N} l_i \bar{l}_i \cdot (-1) \cdot \prod_{j \neq i} (\lambda_j - \mu) =$$

$$= \prod_{i=1}^{N} (\lambda_i - \mu) \cdot \sum_{i=1}^{N} \frac{|l_i|^2}{\mu - \lambda_i}. \quad \blacksquare$$

Corollary 2: If eigenvalues of A are $\lambda_1 \geq \dots \geq \lambda_N$ and eigenvalues of its restriction to L are $\mu_1 \geq \dots \geq \mu_{N-1}$, then they interlace:

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \mu_{N-1} \geq \lambda_N$$

Proof: Since conjugations do not change eigenvalues and any Hermitian matrix can be diagonalized, we can assume $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix}$ and use Cor. 1

Then $\mu_1, \dots, \mu_{N-1}$ are $(N-1)$ roots of polynomial

$$f(\mu) = \prod_{i=1}^{N} (\mu - \lambda_i) \cdot \sum_{i=1}^{N} \frac{|e_i|^2}{\mu - \lambda_i}$$

. This polynomial has a zero on each segment $[\lambda_i, \lambda_{i+1}]$ because $f(\mu)$ is continuous and the signs of $f(\lambda_i + \varepsilon)$ and $f(\lambda_{i+1} - \varepsilon)$ are different.

very small

Since the total number of roots of degree $(N-1)$ polynomial is $(N-1)$ and there are $(N-1)$ segments $(\lambda_i, \lambda_{i+1})$, we are done. $\blacksquare$

Remark: Essentially the same argument also shows that $\lambda_k = \lambda_{k+1} = \dots = \lambda_{k+m}$, then $\mu_k = \mu_{k+1} = \dots = \mu_{k+m-1} = \lambda_k$.

Corollary 3: Suppose that A and L are random and independent and the law of A is invariant under conjugations $A \to uAu^*$, $u \in \mathcal{U}(N)$. Then the conditional distribution of eigenvalues $(\mu_i)_{i=1}^{N-1}$ of $A|_L$ given the eigenvalues $(\lambda_i)_{i=1}^{N}$ of A is the same as $(N-1)$ zeros of

$$\sum_{i=1}^{N} \frac{\xi_i}{\mu - \lambda_i} = 0 \, , \quad \text{where } \xi_i \text{ are i.i.d. } \chi_2^2 = \underbrace{(N(0,1))^2 + (N(0,1))^2}_{\text{independent}}$$

Exercise 1: Show that χ_2^2, defined through, is exponential distribution of mean $\mathbb{E}\chi_2^2 = 2$.

Proof of Corollary 3: We condition on L and on eigenvalues of A. Then L is fixed and deterministic, while A is a uniformly random matrix with fixed deterministic eigenvalues $(\lambda_1, \dots, \lambda_N)$: $A = u \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} u^*$, where u is a uniformly random unitary matrix $u \in \mathcal{U}(N)$

Let P be orthogonal projector onto hyperplane L.
Then we are interested in eigenvalues of $P u \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} u^* P$, which are the same as eigenvalues of $(u^* P u) \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} (u^* P u)$.

$u^* P u$ is orthogonal projector on $u^* L$, which is a uniformly random hyperplane in $\mathbb{C}^N$, since u^* is a uniformly random rotation.

Hence, we can use Corollary 1: if we choose a random vector $(e_1, \dots e_N)$ so that its hyperplane is uniform, then the desired law is the one of roots of $\sum_{i=1}^{N} \frac{|e_i|^2}{\mu - \lambda_i} = 0$

Clearly, the hyperplane would be uniform, if $(e_1, \dots e_N)$ were a uniformly random unit vector

Now take N i.i.d. complex Gaussians $z_i \sim N(0,1) + i\, N(0,1)$
$\uparrow$ $\sqrt{-1}$

Then $\displaystyle \ell_i = \frac{z_i}{\sqrt{\sum_{i=1}^N |z_i|^2}}$ is a unit vector, whose law is $\mathcal{U}(N)$-invariant, since the law of $(z_1, \dots z_N)$ also is.

Hence, μ_i solve $\displaystyle \sum_{i=1}^N \frac{1}{\sqrt{\sum_{i=1}^N |z_i|^2}} \cdot \frac{z_i \bar{z}_i}{\mu - \lambda_i} = 0$

Multiplying by $\sqrt{\sum_{i=1}^N |z_i|^2}$ and denoting $\zeta_i = z_i \bar{z}_i$, we are done $\boxed{}$

<u>Remark</u>: If we divide $\displaystyle \sum_i \frac{\zeta_i}{\mu - \lambda_i} = 0$ by $\displaystyle \sum_{i=1}^N \zeta_i$, then we get $\displaystyle \sum_{i=1}^N \frac{w_i}{\mu - \lambda_i} = 0$, where $\displaystyle w_i = \frac{\zeta_i}{\sum_j \zeta_j}$ is such that the vector $(w_1, \dots w_N)$ is uniform on the simplex $w_i \geq 0$, $\sum_i w_i = 1$.

Indeed, the density of $(\zeta_1, \dots \zeta_N)$ is $\frac{1}{2^N} \exp(-2(\zeta_1 + \dots + \zeta_N))$. Hence, conditioning on $\zeta_1 + \dots + \zeta_N$, we get uniform law.

Example: $N=2$. A - uniformly random matrix with e.v. (λ_1, λ_2)
Applying Cor. 3 with $L = \{(x_1, x_2) \mid x_2 = 0\}$, we compute the law of A_{11}. It solves $\frac{\xi_1}{\mu - \lambda_1} + \frac{\xi_2}{\mu - \lambda_2} = 0$, so that
$\mu = \frac{\lambda_1 \xi_2 + \lambda_2 \xi_1}{\xi_1 + \xi_2}$. For i.i.d. exponential ξ_1, ξ_2, conditional law $(\xi_1 | \xi_1 + \xi_2)$ is uniform on $[0, \xi_1 + \xi_2]$. Hence, μ is uniform on $[\lambda_1, \lambda_2]$

Exercise 2: Take $N=2$ and deal with real symmetric, rather than complex Hermitian matrices. Let A be uniformly random with given eigenvalues, i.e. $A = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

uniformly random rotation, $t \sim$ uniform $[-\pi, \pi)$

Compute explicitly the law of A_{11} and compare with previous example

Our next task is to make the result of Cor. 3 more explicit

Proposition 2: Fix $\lambda_1 \geq \dots \geq \lambda_N$ and let $\mu_1 \geq \dots \geq \mu_{N-1}$ be $(N-1)$ roots of $\underbrace{\prod_{i=1}^{N} (\mu - \lambda_i)}$ $\cdot \sum_{i=1}^{N} \frac{w_i}{\mu - \lambda_i} = 0$, where $(w_1, \dots w_N)$ is uniform on simplex $w_i \geq 0$, $\sum_i w_i = 1$.

This factor is important only if some λ_i coincide.

Then in the situation of distinct λ_i, the density of μ_i becomes:

$$\mathbb{1}_{\lambda_1 > \mu_1 > \dots > \mu_{N-1} > \lambda_N} \cdot (N-1)! \cdot \frac{\prod_{1 \leq i < j \leq N-1} (\mu_i - \mu_j)}{\prod_{1 \leq i < j \leq N} (\lambda_i - \lambda_j)} \, d\mu_1 \cdot \ldots \cdot d\mu_{N-1}$$

And the case $\lambda_i = \lambda_j$ is obtained by continuity (in space of probability measures)

Remark: Coincides with Week 9, Slide 10, Proposition 1

Proof of proposition 2: We have a correspondence between vectors

$$(w_1, \dots w_N \mid \sum w_i = 1, w_i > 0) \longleftrightarrow (\mu_1, \dots \mu_{N-1} \mid \lambda_1 > \mu_1 > \dots > \mu_{N-1} > \lambda_N)$$

and we need to compute the Jacobian of this transformation

By definition of the correspondence:

$$(\Delta) \qquad \sum_{i=1}^{N} \frac{w_i}{\mu-\lambda_i} = \frac{\prod_{i=1}^{N-1}(\mu-\mu_i)}{\prod_{i=1}^{N}(\mu-\lambda_i)}$$

polynomial with roots μ_i and leading coef. 1

Given $(\mu_1,\dots,\mu_{N-1})$, w_i are coefficients of the simple fractions decomposition of the function $R(\mu)$ in the right-hand side of (Δ)

They are found multplying Δ by $\mu-\lambda_i$ and plugging $\mu=\lambda_i$

$$w_i = \frac{\prod_{j=1}^{N-1}(\lambda_i-\mu_j)}{\prod_{j\neq i}(\lambda_i-\lambda_j)}$$

[Note that this is positive because of interlacement of μ_j and λ_j]

For Jacobian we need derivatives:

$$\frac{\partial w_i}{\partial \mu_j} = (-1)\cdot \frac{\overbrace{\prod_{a\neq j}(\lambda_i-\mu_a)}^{(N-2)\text{ factors}}}{\underbrace{\prod_{a\neq i}(\lambda_i-\lambda_a)}_{(N-1)\text{ factors}}}$$

On the simplex $w_i\geq 0$, $\sum_i w_i = 1$ we can choose $(w_1, w_2,\dots, w_{N-1})$ as coordinates for the Lebesgue measure $dw_1\, dw_2 \cdot\ldots\cdot dw_{N-1}$

Therefore, we need to compute

$(N-1)^2$ factors

$$\det\left[\frac{\partial w_i}{\partial \mu_j}\right]_{i,j=1}^{N} = (-1)^N \prod_{\substack{a\neq b\\ a\neq N}} \frac{1}{\lambda_b-\lambda_a} \prod_{a=1}^{N-1}\prod_{b=1}^{N-1}(\lambda_b-\mu_a) \det\left[\frac{1}{\lambda_i-\mu_j}\right]_{i,j=1}^{N-1}$$

Lemma 1 (Cauchy determinant) $\det\left[\frac{1}{x_i-y_j}\right]_{i,j=1}^{K} = \frac{\prod_{1\le i<j\le K}(x_i-x_j)(y_j-y_i)}{\prod_{i=1}^{K}\prod_{j=1}^{K}(x_i-y_j)}$

Proof of Lemma: $\det[\;]\cdot\prod_{i,j}(x_i-y_j)$ is a polynomial of degree $N(N-1)$, which vanishes whenever $x_i=x_j$ or $y_i=y_j$. Hence, it is $\text{const}\cdot\prod_{i<j}(x_i-x_j)\cdot\prod_{i<j}(y_i-y_j)$. Comparing some coefficient we find $\text{const}=(-1)^{\frac{N(N-1)}{2}}$ $\blacksquare$

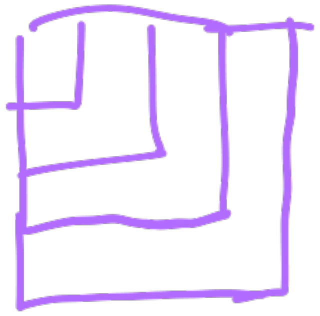
Using the Lemma, we get $\det\left[\frac{dw_i}{\partial\mu_j}\right] = \pm\frac{\prod_{i<j\le N-1}(\mu_i-\mu_j)}{\prod_{i<j\le N}(\lambda_i-\lambda_j)}$

$\frac{1}{\text{Vol(simplex)}}$, which makes total mass $=1$

Hence, $(N-1)!\,dw_1\cdots dw_{N-1} = (N-1)!\,\frac{\prod(\mu_i-\mu_j)}{\prod(\lambda_i-\lambda_j)}\,d\mu_1\cdots d\mu_{N-1}$ $\blacksquare$

Theorem 1: Let A be $N{\times}N$ random [Hermitian] matrix invariant under $A \to uAu^*$, $u \in U(N)$

Let x_i^k, $1 \le i \le k$ be eigenvalues of $k{\times}k$ corner of A



Then:

1) Eigenvalues interlace: $x_i^{k+1} \geq x_i^k \geq x_{i+1}^{k+1}$ for all $1 \le i \le k < N$

2) Conditional law of $(x_1^{N-1}, \dots x_{N-1}^{N-1})$ given $(x_1^N, \dots, x_N^N) = (y_1, \dots y_N)$ is

$$(N-1)!\,\frac{\prod_{i<j}(x_i^{N-1}-x_j^{N-1})}{\prod_{i<j}(y_i-y_j)}\, dx_1^{N-1}\cdot\ldots\cdot dx_{N-1}^{N-1}$$

3) Conditional law of $\{x_i^k\}_{1\le i\le k<N}$ given $(x_1^N, \dots, x_N^N)$ is uniform (subject to 1))

Proof: 1) is Corollary 2, 2) is a combination of Corollary 3 and Proposition 2
3) is obtained from 2) by induction in N. $\boxed{\text{}}$

Тр. МИАН СССР, 1950, том 36, страницы 3–288 (Mi tm1100)

Унитарные представления классических групп

И. М. Гельфанд, М. А. Наймарк

Origin of ideas

Rayleigh triangles and non-matrix interpolation of matrix beta integrals

Yu A Neretin[1]

Sbornik: Mathematics, Volume 194, Number 4

Important generalizations

GUEs and queues

Yu. Baryshnikov

We followed this text

Probability Theory and Related Fields **119**, 256–274 (2001)

We have established a correspondence

$\{$coherent systems on $\mathbb{S}p\}$ $\longleftrightarrow$ $\{$random $\mathcal{U}(\infty)$-invariant Hermitian matrices$\}$

But how do we classify either of these objects?

The Martin boundary question is rephrased as:

Take $\lambda = \lambda(N) \in \mathbb{S}p_N$ and let $A_N = u \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} u^*$

[$u \in \mathcal{U}(N)$ -uniformly random]

How should λ depend on N for existence of distributional

$$\lim_{N\to\infty} \left[k\times k \text{ corner of } A_N \right] = ?$$

For $k=1$ answering this question is quite easy

Proposition 3: Let $A_N = [a_{ij}]_{i,j=1}^N$ be uniformly random Hermitian matrix with eigenvalues $\lambda_1 \geq \dots \geq \lambda_N$. Then

$$a_{11} \overset{d}{=} \sum_{i=1}^N w_i \lambda_i ,$$ where $(w_1, \dots w_N)$ is uniformly random point in the simplex $w_i > 0$, $\sum_{i=1}^N w_i = 1$

Proof. $$a_{11} = \left[U \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} U^* \right]_{11} = \sum_{i=1}^N U_{1i} \lambda_i \overline{U_{1i}} = \sum_{i=1}^N |U_{1i}|^2 \lambda_i$$

The first row of uniformly random matrix U is a uniformly random unit vector $(U_{11}, U_{12}, \dots, U_{1N})$: $\sum_{i=1}^N |U_{1i}|^2 = 1$

As we saw on Slide 8, therefore, $(|U_{11}|^2, \dots, |U_{1N}|^2) \overset{d}{=} (w_1, \dots w_N)$ $\blacksquare$

Using Slide 8 again, an alternative form is

$$a_{11} \overset{d}{=} \frac{2N}{\sum_{i=1}^N \xi_i} \sum_{i=1}^N \frac{\xi_i}{2} \cdot \frac{\lambda_i}{N} \quad (**)$$ where ξ_i are i.i.d. exponential of mean 2

Proposition 4: Suppose that $\lambda(N) \in Sp_N$ is such that the 1×1 corner of uniformly random matrix with eigenvalues $\lambda(N)$ converges in distribution to a random variable η. Then

should be finite

$$\lim_{N\to\infty} \frac{\lambda(N)_i}{N} = \alpha_i^+ ; \quad \lim_{N\to\infty} \frac{\lambda(N)_{N-i}}{N} = \alpha_i^- ; \quad \lim_{N\to\infty} \frac{\sum_{i=1}^N \lambda(N)_j}{N} = \gamma_1 ; \quad \lim_{N\to\infty} \sum_{i=1}^N \left(\frac{\lambda(N)}{N}\right)^2 = \gamma_2 + \sum_i (\alpha_i)^2$$

$\{\alpha_i\} = \{\alpha_i^+\} \cup \{\alpha_i^-\}$ and

$$\mathbb{E} e^{it\eta} = e^{it\gamma_1} e^{-\gamma_2 \frac{t^2}{2}} \prod_{j\geq 1} \frac{e^{-it\alpha_j}}{1 - it\alpha_j}$$

$\sqrt{-1}$

Proof. Using the formula from the previous slide and noting that $\lim_{N\to\infty} \frac{2N}{\sum_{i=1}^N \xi_i} = 1$ (a.s.) by Strong Law of Large Numbers,

$$\eta = \underbrace{\lim_{N\to\infty} \sum_{i=1}^N \frac{\xi_i}{2} \cdot \frac{\lambda(N)_i}{N}}_{\eta(N)}.$$

Convergence in distribution is equivalent to convergence of characteristic functions.

Hence, denoting $\eta(N)$, we need to study

$$\lim_{N\to\infty} e^{it\eta(N)} = \lim_{N\to\infty} \prod_{j=1}^{N} \frac{1}{1-it\frac{\lambda(N)_j}{N}}$$

Where we used characteristic function of exponential random variable and multiplied over independent summands.

Comparing with Week 2, Slide 8, we see the match, if we invert function, replace $z=it$, rename $a_j^N = \lambda(N)_j$.

Hence, Week 2, Slides 8-11 give the desired statement $\blacksquare$

This proposition settles the question of laws of 1×1 corners. How do we proceed to $K\times K$ corners for arbitrary K? There are several approaches in the literature:

I) Develop formulas for $K\times K$ corner of $N\times N$ matrix [extending prop. 3]

II) Find functional relation 1×1 corner $\rightarrow$ $K\times K$ corner of $U(N)$-invariant $N\times N$ matrix

III) Characteristic functions of $U(\infty)$-invariant ergodic matrices are always multiplicative.

Theorem 2: Take a random $\infty \times \infty$ Hermitian matrix A, whose law is $U(\infty)$-invariant. Then A is ergodic (= its law is extreme) if and only if its characteristic function factorizes:

$$\mathbb{E}\, e^{i \operatorname{Trace}(XA)} = \prod_{\substack{\text{eigenvalues} \\ \text{of } X \\ (x_j)}} f(x_j), \quad \text{where}$$

infinite Hermitian matrix with finitely many non-zero matrix elements

f is some continuous function satisfying $f(0) = 1$.

[it does not matter, whether you include $x_j = 0$ or not]

The proof of this theorem is based on:

Lemma 2: Suppose that A_N is $N \times N$ random $U(N)$-invariant Hermitian matrix with fixed (deterministic) spectrum. Then denoting $\varphi(X) := \mathbb{E}_{A_N} e^{i \operatorname{Trace}(XA_N)}$ for any $N \times N$ Hermitian X and Y we have:

$$\int_{U(N)} \varphi(X + uYu^*)\, du = \varphi(X)\, \varphi(Y)$$

Proof of Lemma: First, note that for $A_n = [a_{ij}]_{i,j=1}^N$, $X = [x_{ij}]_{i,j=1}^N$

$\mathrm{Trace}(XA_n) = \sum_{i,j=1}^N x_{ij} a_{ji}$. Hence, $\mathbb{E}\, e^{i\,\mathrm{Trace}(XA)}$ is the usual characteristic function of A_N as a random vector

[In vector space of $N \times N$ Hermitian matrices of (real) dimension N^2]

Next, we can represent $A_n = W \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} W^*$, $W \in \mathcal{U}(N)$ uniformly random

We can also write $X = Q \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix} Q^*$, $Q \in \mathcal{U}(N)$ deterministic

$\mathrm{Trace}(XA_N) = \mathrm{Trace}\left(Q \begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_n \end{pmatrix} Q^* W \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} W^*\right) = \mathrm{Trace}\left(\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix} V \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} V^*\right)$

where $V = Q^* W$ is again a uniformly random element of $\mathcal{U}(N)$

Hence, $\varphi(X) = \mathbb{E}_V\, e^{i\,\mathrm{Trace}\left(\begin{pmatrix} x_1 & & 0 \\ & \ddots & \\ 0 & & x_N \end{pmatrix} V \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_N \end{pmatrix} V^*\right)}$ is actually a function of (x_i) - e.v. of X, and (λ_i) - e.v. of A_n, invariant under $(x_i) \leftrightarrow (\lambda_i)$

Now take independent A_N, B_N with e.v. (λ_i), (μ_i). Then

$\mathbb{E}\, e^{i\,\mathrm{Trace}(X(A_N+B_N))} = \mathbb{E}\, e^{i\,\mathrm{Trace}(XA_n)} \cdot \mathbb{E}\, e^{i\,\mathrm{Trace}(XB_n)}$ by independence

Which can be rewritten as

$$\int_{w,v\in\mathcal{U}(N)} e^{i\,\mathrm{Trace}\left(\begin{pmatrix}x_1 & & 0\\ & \ddots & \\ 0 & & x_N\end{pmatrix}\left(w\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{pmatrix}w^* + v\begin{pmatrix}\mu_1 & & 0\\ & \ddots & \\ 0 & & \mu_N\end{pmatrix}v^*\right)\right)} dw\,dv$$

$$= \int_{w\in\mathcal{U}(N)} e^{i\,\mathrm{Trace}\left(\begin{pmatrix}x_1 & & 0\\ & \ddots & \\ 0 & & x_N\end{pmatrix}w\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_N\end{pmatrix}w^*\right)} dw \cdot \int_{v\in\mathcal{U}(N)} e^{i\,\mathrm{Trace}\left(\begin{pmatrix}x_1 & & 0\\ & \ddots & \\ 0 & & x_N\end{pmatrix}v\begin{pmatrix}\mu_1 & & 0\\ & \ddots & \\ 0 & & \mu_N\end{pmatrix}v^*\right)}$$

If we write $v = wu$, then we get precisely the desired statement written in renamed variables. $\blacksquare$

<u>Sketch of the proof of Theorem 2:</u> Let $\varphi(X) = \mathbb{E}_A\, e^{i\,\mathrm{Trace}(XA)}$ and assume that A is ergodic. Note $\varphi(uXu^*) = \varphi(X)$, $u\in\mathcal{U}(\infty)$, as follows from $\mathcal{U}(\infty)$-invariance of A. Hence, multiplicativity reduces to showing that for any two finite matrices X and Y

$$\varphi\begin{pmatrix}X & 0\\ 0 & Y\end{pmatrix} = \varphi(X)\cdot\varphi(Y)$$

We claim that this statement is $N\to\infty$ limit of Lemma 2

Indeed, let X and Y be $n\times n$ [padded with zeros to larger sizes, if necessary]

Let $U\in\mathcal{U}(N)$ be uniformly random. In block form $U=\begin{array}{cc|l} \begin{pmatrix} A & B \\ C & D \end{pmatrix} & & \begin{array}{l} \updownarrow n \\ \updownarrow N-n \end{array} \\ \overset{\longleftrightarrow}{n} \quad \overset{\longleftrightarrow}{N-n} & & \end{array}$

$$U\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}U^* = \begin{pmatrix} AY & 0 \\ CY & 0 \end{pmatrix}\begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} = \begin{pmatrix} AYA^* & AYC^* \\ CYA^* & CYC^* \end{pmatrix}$$

Now if $N \gg n$, then $\|A\|$ is very small, because

$A^*A + C^*C = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$ $\updownarrow n$ (size $n\times n$) by orthogonality and C is a much larger matrix [This step needs more details for the full proof]

Therefore, $U\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix}U^* \approx \begin{pmatrix} 0 & 0 \\ 0 & CYC^* \end{pmatrix}$ and nonzero eigenvalues (CYC^*) are almost the same as those of Y

Hence, e.v. $(X+UYU^*) \approx$ e.v. $\begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}$. Hence, using Lemma 2 and sending $N\to\infty$, we conclude that each measure from Martin boundary (thus, also from minimal boundary) is multiplicative. This proves one direction.

Multiplicativity ⇒ extremality: Choosing X to be diagonal, we see that diagonal elements A_{kk} are i.i.d.. Thus, by using SLLN for $\mathbb{1}_{A_{kk}\in[a,b]}$, we can reconstruct the law of A_{11}, and, thus, $f(\cdot)$ from observing all A_{kk}. This means that distinct f lead to measures with disjoint supports. ☒

We now have the full proof for Week 9, Slide 15, Theorem 5:

[Boundary of the graph of spectra and LLN for ergodic measures]

By Proposition 4 and embedding (minimal boundary) ⊂ (Martin boundary) 1×1 corners have the form of Week 9, Theorem 5. By Theorem 2 then all $k\times k$ corners have the form of Week 9, Theorem 5. In addition, the same Theorem 2 also shows that all measures of Week 9, Theorem 5 are ergodic.

By following the argument of Week 7, Slides 6-8, Proposition 4 combined with classification of extreme measures implies the LLN for these measures. ☒

We end by mentioning another link to applied mathematics.

Definition: Fix N and $t_1 > t_2 > \ldots > t_N$ called **Knots**.
Fundamental spline (or B-spline) is a unique function $M(x\,;\,t_1,\ldots,t_N)$ of real argument $x \in \mathbb{R}$, such that:

- In each interval (t_{i+1}, t_i), $M(x)$ is a polynomial of degree $N-2$
- $M(x) = 0$ for $x < t_N$ and $x > t_1$
- $M(x)$ has $(N-3)$ continuous derivatives at each Knot
- $\int_{-\infty}^{\infty} M(x)\,dx = 1$

Proposition 5: $M(x; t_1, \ldots, t_N)$ coincides with the density of 1×1 corner of uniformly random $N \times N$ Hermitian matrix A with eigenvalues $(t_1, t_2, \ldots t_N)$

References for further reading:

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