Math 833 — Degenerations of the Gelfand-Tsetlin graph — Week 9

Today we connect GT-graph to Young graph and to random matrices through two limit transitions

Gelfand-Tsetlin graph

Young bouquet

A close relative of the Young graph

Graph of spectra

Describes eigenvalues of random $U(N)$-invariant Hermitian matrices

We start from the Young graph.

Comparing Week 5, Theorem 2 with Week 8, Theorem 2 we see a remarkable similarity: the boundaries of $\mathbb{GT}$ and $\mathbb{Y}$ are described by almost identical formulas.

But why? The branching rules in these graphs look very different!

From representation-theoretic point of view, we discuss unitary groups $U(N)$ and symmetric groups $S(n)$.

Despite their different natures, the theories have similarities

- For both the character theory links to Schur functions
- Construction of bases and matrix realisations of irreducible reps can be done in a similar way (Gelfand-Tsetlin basis vs Young's orthogonal form)
- Both theories can be constructed from the same block: Schur-Weyl duality

Take two integers $N, n > 0$ and look at

$$(\mathbb{C}^N)^{\otimes n}$$

This is a linear space of dimension N^n

$\mathcal{U}(N)$ acts here: $u(v_1 \otimes v_2 \otimes \dots \otimes v_n) := (uv_1) \otimes (uv_2) \dots \otimes (uv_n)$

$S(n)$ also acts here: $\sigma(v_1 \otimes \dots v_n) := v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(n)}$

This actions commute: $u \cdot \sigma (v_1 \otimes \dots \otimes v_n) = \sigma \cdot u (v_1 \otimes \dots \otimes v_n)$

Theorem 1: (Schur-Weyl duality)

irred. rep of $S(n)$

$$(\mathbb{C}^N)^{\otimes n} = \bigoplus_{\lambda \in \mathbb{Y}_n \cap \mathbb{GT}_N} T_\lambda^{S(n)} \otimes T_\lambda^{\mathcal{U}(N)}$$

irred. rep of $\mathcal{U}(N)$

in particular, $N^n = \sum \dim(\lambda) \cdot \mathrm{Dim}_N(\lambda)$

dimension in $\mathbb{Y}$ dimension in $\mathbb{GT}$

We are not proving Theorem 1. Its important conceptual consequence is that irreducible representations of either $\mathcal{U}(N)$ or $S(n)$ can be all constructed decomposing the same vector space $(\mathbb{C}^N)^{\otimes n}$.

[For $\mathcal{U}(N)$ one also needs to be able to shift all coordinates to get negative λ_i, but this can be achieved by tensoring with 1-dimensional representations $u \to \det(u)^k$]

Hence, there is a link $\mathcal{U}(N) \leftrightarrow S(n)$ already on finite level.

At $N=n=\infty$, the link is even tighter

Take $\lambda \in \mathbb{Y}$ and large N, identify λ with an element of GT_N $(\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N)$ [has many 0's]

Question: How does a uniformly random path $\emptyset \to \lambda$ in GT look like?

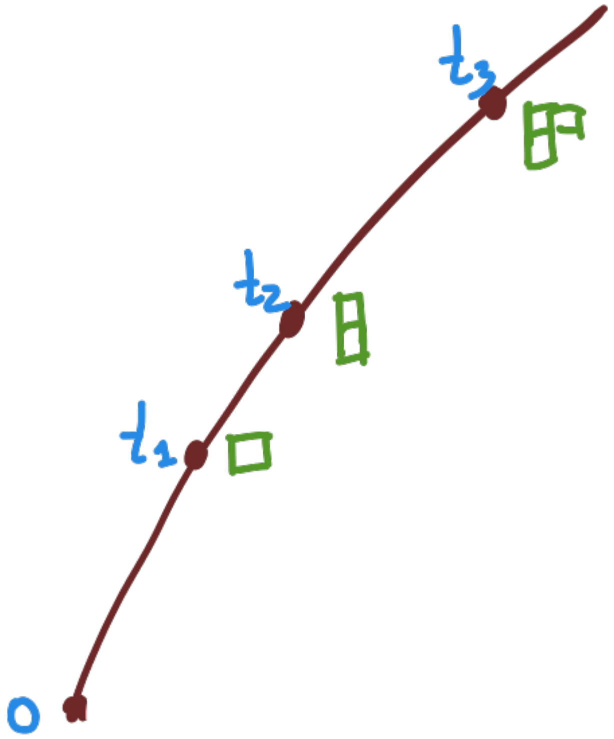
Answer: This is a sequence of Young diagrams $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \lambda^{(3)} \subset \dots \subset \lambda^{(N)} = \lambda$, in which for most indices K we have $\lambda^{(K)} = \lambda^{(K+1)}$ and only for $n = |\lambda|$ of those we have $\lambda^{(K+1)} = \lambda^{(K)} + \square$

[$\lambda^{(K+1)} \succ \lambda^{(K)}$ guarantees $\lambda^{(K)} \subset \lambda^{(K+1)}$. In principle, we could have $|\lambda^{(K+1)} / \lambda^{(K)}| \geq 2$, but probability of this event $\to 0$ as $N \to \infty$]

Conclusion: Positive part of GT (signatures with $\lambda_N \geq 0$) has a limit "small diagrams, large N", in which we get the Young graph with an upgrade: boxes are added to a Young diagram not at every step, but at random times

The upgraded Young graph is called the Young bouquet

Paths in $\mathbb{YB}$ have the form



t_k = time when the k-th box is added

Stem with leaves (= Young diagrams).
Hence, the name Young bouquet

Definition: $\mathbb{YB}$ is a branching graph with $\mathbb{R}_{\geq 0}$ grading.
The r-th level of $\mathbb{YB}$ for $r \geq 0$ [real number!] is the countable set of pairs (λ, r) [r is fixed in $\mathbb{YB}_r$]
λ - arbitrary Young diagram

The cotransition probabilities are defined for $r' < r$ by

$$(\lambda, r) \to (\mu, r') \qquad \left(\frac{r'}{r}\right)^m \left(1 - \frac{r'}{r}\right)^{n-m} \frac{n!}{m!\,(n-m)!} \cdot \frac{\dim \mu \cdot \dim \lambda/\mu}{\dim \lambda}$$

$|\lambda| = n$, $|\mu| = m$, $\mu \subset \lambda$

Binomial distribution

Links in Young graph

In words, we transition from (r,λ) to level $r'<r$ in 2 steps

- Choose m from Binomial distribution on $\{0,1,\dots,n\}$ with param. $\frac{r'}{r}$
- Choose μ with $|\mu|=m$ according to links $L_{n\to m}(\lambda\to\mu)$ from the Young graph.

Althoug the grading is now not by integers, but by real numbers, we can still define coherent systems and boundary

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Definition: M_r, $r\geq 0$ is a coherent system on $\mathbb{YB}$ if each M_r is a probability measure on Young diagrams λ and

$$M_{r'}(\mu)=\sum_{\lambda\in\mathbb{Y}}\left(\frac{r'}{r}\right)^{|\mu|}\left(1-\frac{r'}{r}\right)^{|\lambda|-|\mu|}\frac{|\lambda|!}{|\mu|!(|\lambda|-|\mu|)!}\cdot\frac{\dim\mu\cdot\dim\lambda/\mu}{\dim\lambda}\cdot M_r(\lambda)$$

for any $r'<r$ and $\mu\in\mathbb{Y}$

Theorem 2: The extreme coherent systems on $\mathbb{YB}$ are parameterized by $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$, $\beta_1 \geq \dots \geq 0$ with $\sum_i (\alpha_i + \beta_i) \leq 1$ and $x \geq 0$ [when $x=0$, α_i and β_i parameters are not needed].

Explicitly they are given by:

$$M_r^{(\alpha,\beta;x)}(\lambda) = e^{-rx} \frac{(rx)^{|\lambda|}}{|\lambda|!} \cdot M_{|\lambda|}^{(\alpha,\beta)}(\lambda)$$

extreme coherent system on Young graph

In words coherent systems on $\mathbb{YB}$ are Poissonizations of ones on $\mathbb{Y}$

Theorem 3:

$$M_r^{(\alpha,\beta;x)}(\lambda) = \lim_{N\to\infty} M_N^{\left(\alpha^+ = \frac{x}{N}\alpha,\ \beta^+ = \frac{x}{N}\beta,\ \gamma^+ = \frac{x}{N}\left(1-\sum_i(\alpha_i+\beta_i)\right)\right)}(\lambda)$$

extreme coherent system on Gelfand-Tsetlin graph

and

$$\left(\frac{r'}{r}\right)^{|\mu|}\left(1-\frac{r'}{r}\right)^{|\lambda|-|\mu|} \frac{|\lambda|!}{|\mu|!\,(|\lambda|-|\mu|)!} \cdot \frac{\dim \mu \cdot \dim \lambda/\mu}{\dim \lambda} = \lim_{N\to\infty} L_{Nr \to Nr'}(\lambda \to \mu)$$

link in GT

We will not give proofs of Theorems 2 and 3
[They are not hard, feel free to figure out the proofs yourself]

They were established in:

Moscow Mathematical Journal

Volume 13, Issue 2, April–June 2013 pp. 193–232.

The Young Bouquet and its Boundary

Authors: Alexei Borodin (1) and Grigori Olshanski (2)

Example: Through Theorem 3 the measures $M_N^{(\gamma^+ = t/N)}$ on $\mathbb{GT}$
[Connected to TASEP, see slides 14-16 of Week 8]
converge as $N \to \infty$ to Poissonized Plancherel measure
on Young diagrams with $\mathrm{Prob}(\lambda) = e^{-t} \dfrac{t^{|\lambda|}}{(|\lambda|!)^2} \cdot [\dim(\lambda)]^2$

dimension in the Young graph

For the second continuous limit of GT we keep N fixed and make $\lambda_1 \geq \dots \geq \lambda_N$ very large

Proposition 1: Take real numbers $x_1 > y_1 > x_2 > \dots > x_N > y_N > x_{N+1}$

integer part

Then $\lim_{\varepsilon\to 0} \varepsilon^{-N} L_{(N+1)\to N}\left(\lfloor \varepsilon^{-1}x_1 \rfloor, \dots, \lfloor \varepsilon^{-1}x_{N+1} \rfloor \to \lfloor \varepsilon^{-1}y_1 \rfloor, \dots, \lfloor \varepsilon^{-1}y_N \rfloor\right)$

Cotransitional probability (link) in Gelfand-Tsetlin graph

$$= N! \cdot \frac{\prod_{i<j\leq N}(y_i - y_j)}{\prod_{i<j\leq N+1}(x_i - x_j)}$$

Proof. By Week 8, Slide 3

$$L_{(N+1)\to N}(\lambda, \mu) = N! \cdot \frac{\prod_{i<j\leq N}(\mu_i - i - (\mu_j - j))}{\prod_{i<j\leq N+1}(\lambda_i - i - (\lambda_j - j))}.$$

Remains to take direct limit

$\blacksquare$

Definition: The graph of spectra $\mathbb{Sp}$ has levels

$$\mathbb{Sp}_N = \{ x_1 \geq x_2 \geq \dots \geq x_N \mid x_i \in \mathbb{R} \}$$ and

cotransition probabilities $L_{(N+1)\to N}\big((x_1, \dots x_{N+1}) \to (y_1, \dots y_N)\big)$

given by
$$\begin{cases} N! \cdot \dfrac{\prod (y_i - y_j)}{\prod (x_i - x_j)} \, dy_1 \, dy_2 \cdot \ldots \cdot dy_N, & \text{if } x_1 > y_1 > x_2 > \dots > y_N > x_{N+1} \\ 0, & \text{otherwise.} \end{cases}$$

[This is for the case $x_1 > \dots > x_{N+1}$. If some coordinates coincide, then we extend the definition by continuity in the space of measure. For instance, if $x_1 = \dots = x_{N+1} = 0$, then we get the unit mass (δ-measure) at $y_1 = \dots = y_N = 0$.]

Same as $\mathbb{GT}$, but with integers replaced by reals.

Paths in the graph of spectra are interlacing arrays of reals

$$\begin{array}{ccccccc} & & & x_1^1 & & & \\ & & x_1^2 & & x_2^2 & & \\ & x_1^3 & & x_2^3 & & x_3^3 & \\ x_1^4 & & x_2^4 & & x_3^4 & & x_4^4 \end{array}$$

By taking the limit from Gelfand-Tsetlin graph, the centrality property (= coherency) says that:

- Given $(x_1^N \geq x_2^N \geq \dots \geq x_N^N) \in Sp_N$ the conditional law of subarray $\{x_i^j\}_{1 \leq i \leq j \leq N-1}$ is uniform on the polytope defined by interlacing inequalities

- The dimension in Sp is the volume of this polytope

$$\mathrm{Dim}_N(x_1, \dots x_N) = \mathrm{Volume} = \prod_{1 \leq i < j \leq N} \frac{x_i - x_j}{j - i}$$

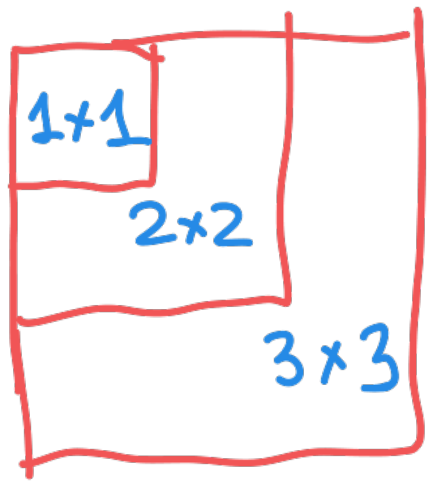
[This is the limit of Theorem 1 on slide 3 of week 8]

Why is this called the graph of spectra?

Theorem 4: Take $N \times N$ random Hermitian matrix, whose law is invariant under conjugations
$$A \to U A U^*, \quad U \in \mathbb{U}(N).$$

[Conjugations preserve being Hermitian and eigenvalues. It only changes eigenvectors. Hence, in conjugation-invariant matrix, the eigenvectors are uniformly chosen (conditional on eigenvalues)]

Let x_i^k be the i th largest eigenvalue of top-left $k \times k$ corner



Then:

- Eigenvalues interlace: $x_i^{k+1} \geq x_i^k \geq x_{i+1}^{k+1}$ for all $1 \leq i \leq k < N$
- Given $(x_1^N, x_2^N, \dots x_N^N)$ the conditional law of $\{x_i^k\}_{1 \leq i \leq k < N}$ is uniform on the polytope of interlacing inequalities

Proof of Theorem 4 — next week

Using Theorem 4 the coherent systems on $\mathbb{S}p$ are random infinite Hermitian matrices invariant under conjugations by $u \in \mathcal{U}(\infty) = \bigcup_{N\geq 1} \mathcal{U}(N)$

The bijection is as follows:

infinite Hermitian matrix A

$$\Updownarrow$$

laws of all its corners A_N

$$\Updownarrow$$

laws of all eigenvalues $(x_1^N \geq x_2^N \geq \dots \geq x_N^N)$ of A_N

Hence, boundary of $\mathbb{S}p$ is in bijection with laws of ergodic $\mathcal{U}(\infty)$-invariant random $\infty \times \infty$ Hermitian matrices.

Theorem 5: Extreme coherent systems on Sp are parameterized by $\gamma_1 \in \mathbb{R}$, $\gamma_2 \geq 0$, (α_j) with $\sum (\alpha_j)^2 < \infty$

The projection on the first level, $\mu_1^{(\alpha_i;\gamma_1,\gamma_2)}(x)$ has characteristic function

$\sqrt{-1}$ shift Gaussian random variable Shifted exponential random variable

$$\mathbb{E}_{\mu_1^{(\alpha_j;\gamma_1,\gamma_2)}} e^{itx} = \varphi^{(\alpha_j;\gamma_1,\gamma_2)}(t) = e^{i\gamma_1 t} e^{-\gamma_2 \frac{t^2}{2}} \prod_{j \geq 1} \frac{e^{-it\alpha_j}}{1 - it\alpha_j}$$

More generally, the random matrix A_N has the law such that

$$\mathbb{E} \exp\left(i \operatorname{Trace}\left(T A_N\right)\right) = \varphi^{(\alpha_j,\gamma_1,\gamma_2)}(t_1)\, \varphi^{(\alpha_j,\gamma_1,\gamma_2)}(t_2) \cdots \varphi^{(\alpha_j,\gamma_1,\gamma_2)}(t_N)$$

arbitrary Hermitian matrix eigenvalues of T

Further, denoting $(x_1^N \geq x_2^N \geq \ldots \geq x_N^N)$ the eigenvalues of A_N, the LLN holds: $\lim_{N\to\infty} \frac{x_j^N}{N} = \alpha_j^+$; $\lim_{N\to\infty} \frac{x_{N-j}^N}{N} = \alpha_j^-$; $\{\alpha_j\} = \{\alpha_j^+\} \cup \{\alpha_j^-\}$; $\lim_{N\to\infty} \frac{x_1^N + x_2^N + \ldots + x_N^N}{N} = \gamma_1$; $\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{x_i^N}{N}\right)^2 = \gamma_2 + \sum_{j=1}^{\infty} (\alpha_j)^2$

Example 1: Fix a (deterministic) number $q \in \mathbb{R}$ and consider a matrix

$$\begin{pmatrix} q & & 0 & \cdots \\ & q & & \\ & & q & \\ 0 & & & q \\ \vdots & & & & \ddots \end{pmatrix}$$

It is invariant under $\mathcal{U}(\infty)$-conjugations and corresponds to measure with $\gamma_1 = q$

Example 2: Let X be infinite matrix with i.i.d. $N(0,1) + i\,N(0,1)$ matrix elements. Consider $\frac{c}{2}(X + X^*)$

$\sqrt{-1}$

Invariance of i.i.d. Gaussian sequences under rotations implies invariance under $\mathcal{U}(\infty)$-conjugations. Corresponds to the measure with $\gamma_2 = c^2$

This is the Gaussian Unitary Ensemble of random matrices

It represents a wider class of Wigner matrices with i.i.d. (modulo symmetry) matrix elements.

Example 3: Take a vector V with i.i.d. $N(0,1)+iN(0,1)$ coordinates and consider the matrix $\frac{a}{2}V^*V$. Due to invariance of the i.i.d. Gaussian sequence under rotations this matrix is invariant under $\mathcal{U}(\infty)$-conjugations. It corresponds to the parameters $\alpha_1 = \gamma_1 = a$.

This is the simplest example of Wishart or sample covariance random matrix. Also sometimes called Laguerre Unitary Ensemble

Exercise: Check that the LLN claimed in Theorem 5 holds for the random matrices of Example 3

Proposition 2: All measures of Theorem 5 are sums of independent matrices from Examples 1, 2, and 3.

Proof: When we add independent random variables, their characteristic functions are multiplied. This is precisely the multiplicative form of the function $\varphi^{(\alpha_j;\gamma_1,\gamma_2)}(t)$.

We prove Theorem 5 next week. For now, some connections:

- γ_1 is the limit of $\gamma^{\pm}$ in the Gelfand-Tsetlin graph. α_i are limits of $\alpha_i^{\pm}$ in GT. However, $\beta^{\pm}$ disappeared and new parameter γ_2 appeared in the limit transition

- $$\left(\Phi^{(\alpha_j;\gamma_1,\gamma_2)}\left(\frac{z}{i}\right)\right)^{-1} = \Phi_{\alpha,\gamma}(z)$$

Theorem 5 — Week 2, Slide 1, Theorem 1

Also compare Proposition 2 with week 2, Slide 19, Theorem 5

An explanation is coming in two week!

- The connection to total positivity, which we saw for Y and GT, still exists for Sp. This time we need to deal with totally positive functions of real argument instead of sequences. They are called Pólya frequency functions.

Conceptual summary:

$$\begin{tikzcd}[column sep=small] & \boxed{\begin{array}{c}\text{Gelfand-Tsetlin graph}\\ \text{discrete } N, \text{ discrete } \lambda_i\end{array}} \arrow[dl, "\begin{array}{c}\text{make } N\\ \text{continuous}\end{array}"'] \arrow[dr, "\begin{array}{c}\text{make } \lambda_i\\ \text{continuous}\end{array}"] & \\ \boxed{\text{Young bouquet}} & & \boxed{\text{Graph of spectra}} \end{tikzcd}$$

Reduction of representation theory of $U(N)$ to the one of $S(n)$

"Semiclassical limit"
Reduction of Lie group $U(N)$ to its Lie algebra (tangent space at identity), which can be identified with (skew)-Hermitian matrices.