Math 833 Gelfand-Tsetlin graph Week 8

So far we dealt with object with $S(n)$ or $S(\infty)$ symmetry:

- Polynomials invariant under permutations of roots
- Exchangeable sequences of random variables
- Representations of $S(\infty)$ (= vector spaces with $S(\infty)$ symmetry)
- Partitions of $\{1,2,\ldots\}$ invariant under permutation

We start dealing with objects with more complicated symmetry governed by unitary groups $\mathcal{U}(N)$ and $\mathcal{U}(\infty)$

$\mathcal{U}(N)$ = compact group of all $N \times N$ complex matrices u satisfying
$$uu^* = u^*u = Id$$

We first define the branching graph and later link to $\mathcal{U}(N)$.

Definition: A signature of rank N is an N-tuple of integers $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N$.

Notation: $GT_N = \{\text{signatures of rank } N\}$

We say that $\lambda \in GT_N$ and $\mu \in GT_{N-1}$ interlace and write $\lambda \succ \mu$, if

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \dots \geq \lambda_{N-1} \geq \mu_{N-1} \geq \lambda_N$$

Definition: Gelfand-Tsetlin graph GT has levels $\{\emptyset\} = GT_0, GT_1, GT_2, \dots$ and edges joining

$\lambda \in GT_N$ and $\mu \in GT_{N-1}$ if and only if $\lambda \succ \mu$

Paths in GT are interlacing arrays

$$
\begin{array}{lccccccccccccc}
\text{Level } 1 & & & & & & & \lambda_1^1 & & & & & & \\
\text{Level } 2 & & & & & \lambda_1^2 & \geq & & \geq & \lambda_2^2 & & & & \\
\text{Level } 3 & & & \lambda_1^3 & \geq & & \geq & \lambda_2^3 & \geq & & \geq & \lambda_3^3 & & \\
\text{Level } 4 & \lambda_1^4 & \geq & & \geq & \lambda_2^4 & \geq & & \geq & \lambda_3^4 & \geq & & \geq & \lambda_4^4
\end{array}
$$

They are called Gelfand-Tsetlin patterns

Theorem 1: $$\mathrm{Dim}_N(\lambda) = \prod_{1 \leq i < j \leq N} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}$$

For the proof see MATH 740, Lecture 4, Slide 11

Exercise: By directly counting paths (not using theorem 1) show that

1) $\mathrm{Dim}_N(k^N) = 1, \quad k \in \mathbb{Z}.$

2) $\mathrm{Dim}_N(k, 0^{N-1}) = \binom{N+k-1}{N-1}, \quad k \in \mathbb{Z}_{\geq 0}.$

Theorem 2: The extremal coherent systems on GT are parameterized by $\alpha_1^+ \geq \alpha_2^+ \geq \dots \geq 0$ $\beta_1^+ \geq \beta_2^+ \geq \dots \geq 0$ $\gamma^+ \geq 0$
$\alpha_1^- \geq \alpha_2^- \geq \dots \geq 0$ $\beta_1^- \geq \beta_2^- \geq \dots \geq 0$ $\gamma^- \geq 0$

subject to $\beta_1^+ + \beta_1^- \leq 1$, $\sum_i (\alpha_i^+ + \beta_i^+ + \alpha_i^- + \beta_i^-) < \infty$

Explicitly they are given by:

$$\mu_N^{(\alpha^\pm, \beta^\pm)}(\lambda) = \mathrm{Dim}_N(\lambda) \cdot \det\left[C_{\lambda_i - i + j}\right]_{i,j=1}^N$$

with

$$\underset{\displaystyle \Phi^{\alpha^\pm, \beta^\pm, \gamma^\pm}(z)}{\underset{\|}{\sum_k c_k z^k}} = e^{\gamma^+(z-1) + \gamma^-(z^{-1}-1)} \prod_{i \geq 1} \frac{1 + \beta_i^+(z-1)}{1 - \alpha_i^+(z-1)} \cdot \frac{1 + \beta_i^-(z^{-1}-1)}{1 - \alpha_i^-(z^{-1}-1)}$$

[Like for the Young graph, but „doubled"]

Remark 1: $\sum_k c_k = \Phi^{\alpha^\pm, \beta^\pm, \gamma^\pm}(1) = 1$

Remark 2: $\beta_1^+ = p$, others $= 0$: essentially i.i.d. Bernoulli(p) coherent system through the embedding of Pascal graph as $(1^k, 0^{N-k}) \in GT_N$

c_k are (two-sided) totally positive sequences found in 50's

for the Young graph they were one-sided ($c_k = 0$, $k < 0$)

On the generating functions of totally positive sequences I

Michael Aissen, I. J. Schoenberg & A. M. Whitney

Journal d'Analyse Mathématique **2**, 93–103(1952) | Cite this article

On the generating functions of totally positive sequences II

Albert Edrei

Journal d'Analyse Mathématique **2**, 104–109(1952) | Cite this article

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C. R. Acad. Sc. Paris, t. 279 (23 décembre 1974) Série A — 945

ANALYSE FONCTIONNELLE. — *Sur les représentations factorielles finies de U (∞) et autres groupes semblables.* Note (*) de M. **Dan Voiculescu**, présentée par M. Jean Leray.

On montre, pour une classe de groupe comprenant U (∞), que les caractères des représentations factorielles finies (type I_n, $n < \infty$ où bien II_1) sont caractérisés parmi les fonctions centrales de type positif par une propriété de multiplicativité et on donne des exemples.

$M_N(\lambda)$ are coefficient of the expansion of an extreme normalized character of $U(\infty)$ into irreducibles when resticted on $U(N)$

$$U(\infty) = \bigcup_{N \geq 1} U(N)$$

Infinite-dimensional unitary group (since 70's)

Coherency relations = branching rules for normalized characters of irreducible representations of $U(N+1)$ restricted on subgroup $U(N)$

$\lambda \in GT_{N+1}$

$\mu \in GT_N$

$$T^\lambda\big|_{U(N)} = \bigoplus_{\mu \prec \lambda} T^\mu$$

[That's what Gelfand and Tsetlin were originally doing in 50's]

V-K found the asymptotic meaning of the parameters

Vershik, A. M.; Kerov, S. V.

Characters and factor representations of the infinite unitary group. (English. Russian original)

Zbl 0524.22017

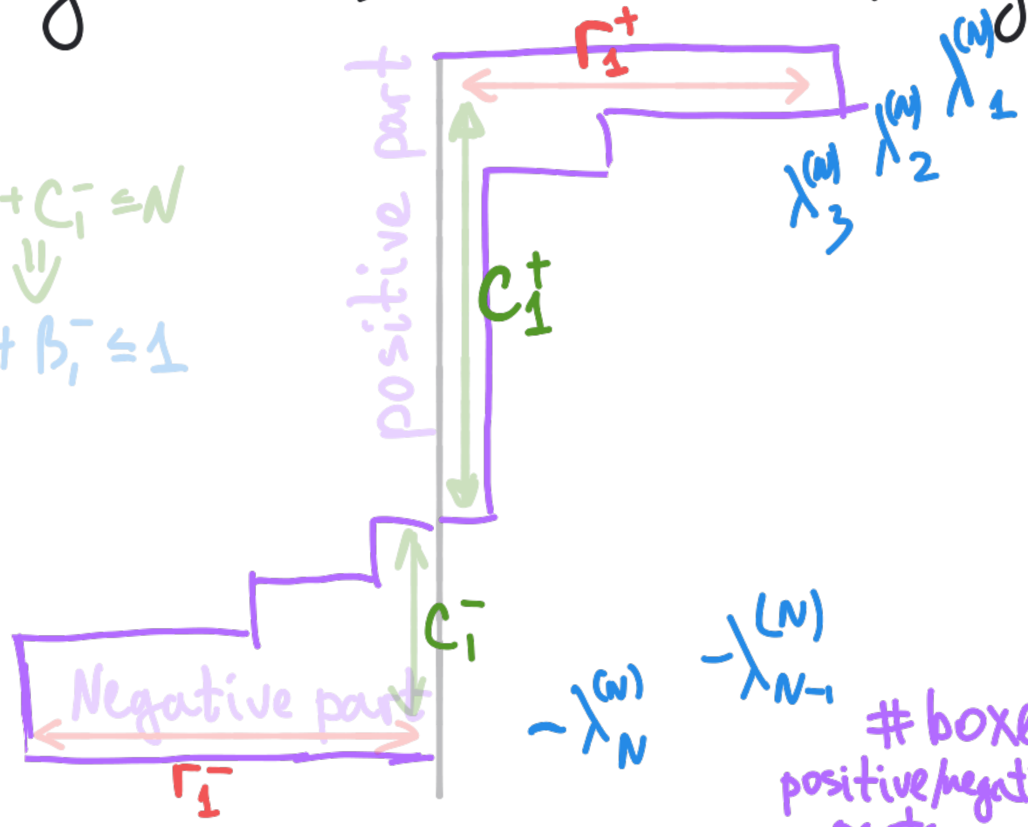
Sov. Math., Dokl. 26, 570-574 (1982); translation from Dokl. Akad. Nauk SSSR 267, 272-276 (1982).

Theorem 3: Let $\lambda^{(N)} \in GT_N$ be a random sequence of signatures distributed by the extreme $(\alpha^\pm, \beta^\pm, \gamma^\pm)$ central measure

$c_1^+ + c_1^- \le N$

$\Downarrow$

$\beta_1^+ + \beta_1^- \le 1$



Then

rows of positive/negative parts

$$\lim_{N\to\infty} \frac{r_i^\pm}{N} = \alpha_i^\pm$$

columns of positive/negative parts

$$\lim_{N\to\infty} \frac{c_i^\pm}{N} = \beta_i^\pm$$

only positive/negative coordinates in the sum

boxes in positive/negative parts

$$\lim_{N\to\infty} \frac{\sum_{i=1}^{N} \left(\lambda_i^{(N)}\right)_\pm}{N} = \pm\,\delta^\pm$$

$$\gamma^\pm + \sum_{i=1}^{\infty} \left(\alpha_i^\pm + \beta_i^\pm\right) = \delta^\pm$$

Modern detailed proofs:

Asymptotics of Jack polynomials as the number of variables goes to infinity

Andrei Okounkov, Grigori Olshanski

International Mathematics Research Notices, Volume 1998, Issue 13, 1998, Pages 641–682,

Extension of Vershik-Kerov approach. Binomial formula for characters

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ELSEVIER

Advances in Mathematics

Volume 230, Issues 4–6, July–August 2012, Pages 1738-1779

MATHEMATICS

The boundary of the Gelfand–Tsetlin graph: A new approach

Alexei Borodin [a, b, c], Grigori Olshanski [d, e]

Moscow Mathematical Journal

Volume 14, Issue 1, January–March 2014 pp. 121–160.

The Boundary of the Gelfand–Tsetlin Graph: New Proof of Borodin–Olshanski's Formula, and its q-Analogue

Authors: Leonid Petrov

Computation of $\mathrm{Dim}(\lambda/\mu) = \#\mathrm{paths}(\mu \to \lambda)$ in GT

The Annals of Probability
2015, Vol. 43, No. 6, 3052–3132
DOI: 10.1214/14-AOP955

ASYMPTOTICS OF SYMMETRIC POLYNOMIALS WITH APPLICATIONS TO STATISTICAL MECHANICS AND REPRESENTATION THEORY

BY VADIM GORIN[1] AND GRETA PANOVA[2]

Contour integral formulas for restrictions of characters of $\mathcal{U}(N)$ on $\mathcal{U}(1)$.

unit circle

A glimpse of the proof.

Theorem 4: Let $\lambda^{(1)} \prec \lambda^{(2)} \prec \ldots \prec \lambda^{(N)} = \lambda$ be a uniformly random path $\varnothing \to \lambda$ in GT. Define $\varphi(z) = \sum_{k \in \mathbb{Z}} z^k \,\mathrm{Prob}(\lambda^{(1)} = k)$

$\lambda^{(1)}$ in GT_1

Then $$\varphi_N^{\lambda}(z) = \frac{s_\lambda(z, 1^{N-1})}{s_\lambda(1^N)}$$ Schur functions, see Week 6, Slide 5

For proving theorem 4 one plugs into $s_\lambda(z_1, \ldots z_N)$ variables $z_i = 1$ and sees interlacing signatures on each step. See Math 740, Lecture 4

Martin boundary: How can $\lambda \in GT_N$ grow with $N \to \infty$, so that $\mathrm{Prob}(\lambda^{(1)} = k)$ converges? $\Longleftrightarrow$ $\exists \lim_{N\to\infty} \varphi_N^\lambda(z)$ uniformly on $z \in \mathbb{C}, |z| = 1$.

A separate argument shows that convergence of $\varphi_N^\lambda(z)$ also implies convergence of all $\mathrm{Prob}(\lambda^{(m)} = \mu)$ for fixed $\mu \in GT_m$

$\lim_{N\to\infty} \varphi_\lambda^N(z)$ is precisely $\varphi^{(\alpha^\pm,\beta^\pm,\gamma^\pm)}$ from Theorem 2

This limit is found by exploiting explicit formulas for $\varphi_\lambda^N(z) = \frac{s_\lambda(z,1^{N-1})}{s_\lambda(1^N)}$ and analyzing their possible limits

Vershik-Kerov and Okounkov-Olshanski use

$$\varphi_\lambda^N(1+x) = \sum_{k\geq 0} h_k^*(\lambda_1,\dots\lambda_N) \cdot \frac{x^k}{N(N+1)\dots(N+k-1)}$$

[See Math 740, Lecture 24, Theorem 2 for a more general formula]

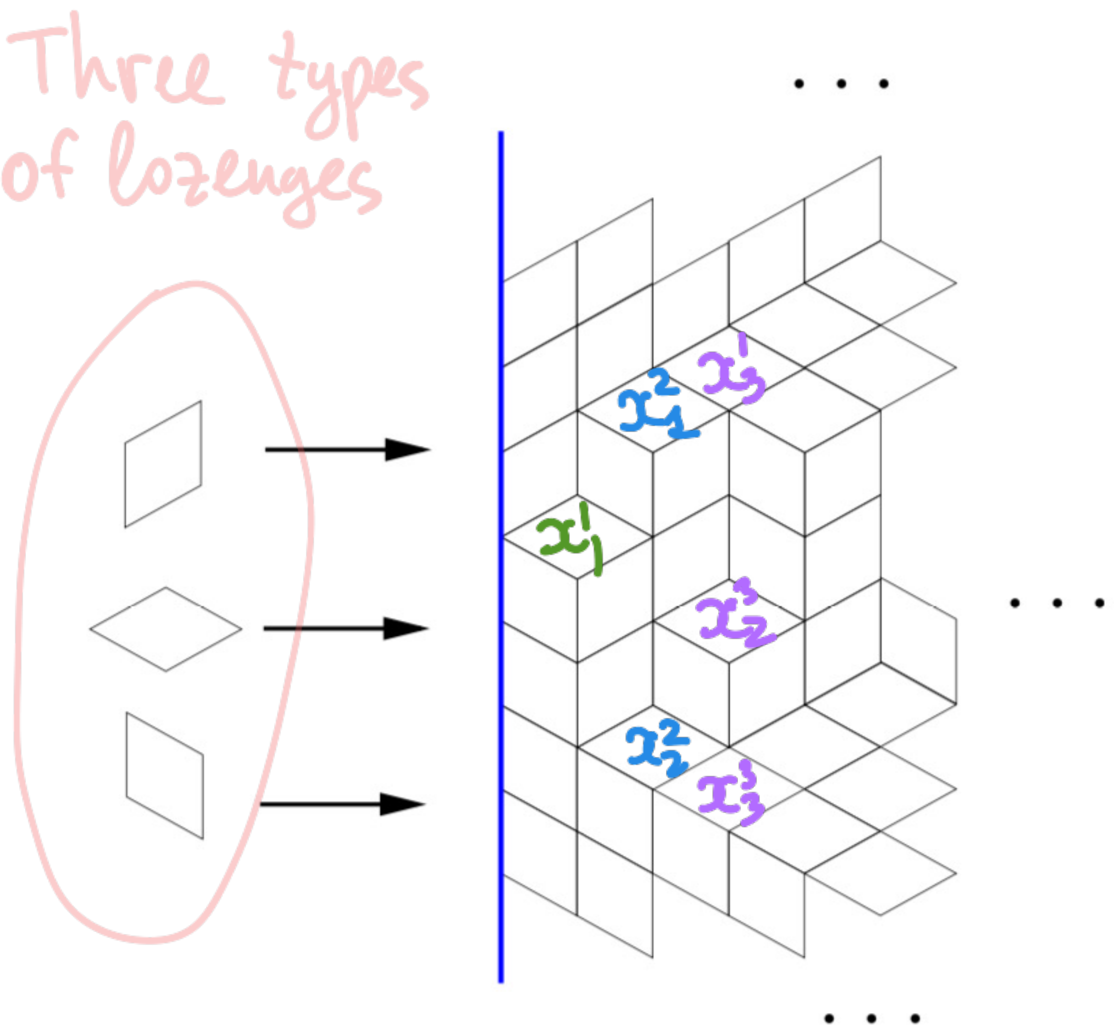
Gorin-Panova instead use

integral over a very large circle ↺

$$\varphi_\lambda^N(z) = \frac{(N-1)!}{(z-1)^{N-1}} \cdot \frac{1}{2\pi i} \oint \frac{z^v}{\prod_{i=1}^N (v-(\lambda_i+N-i))} dv$$

[Expand the determinant in def. of $s_\lambda(z_1,\dots z_N)$ over the row involving z_1 and then plug $z_2=\dots=z_N=1$ into this formula using $s_\mu(1^{N-1})$ evaluation. Get residue expansion of

Remaining lecture: probabilistic faces of Gelfand-Tsetlin graph

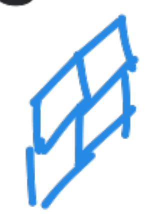
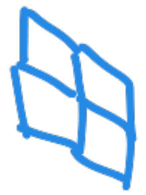


Paths in GT

$\Updownarrow$

Gelfand-Tsetlin patterns

$\Updownarrow$

Lozenge tilings of halfplane, which look  far up and look  far down.

$\lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \dots$

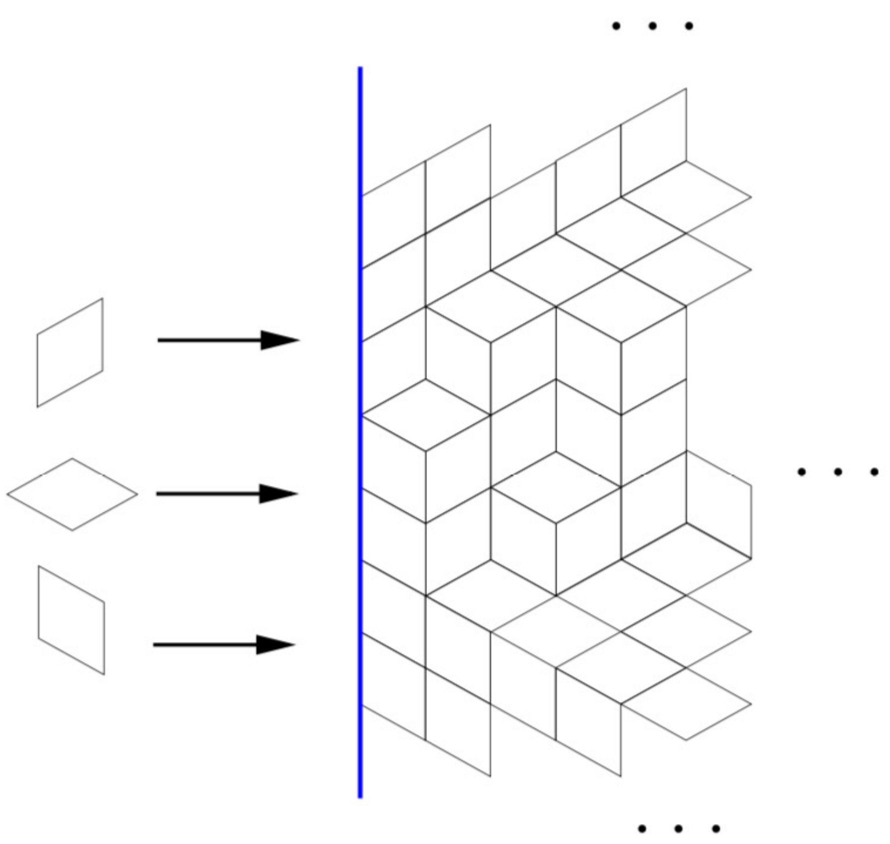
$\lambda^{(N)} = \left(\lambda_1^{(N)} \geq \lambda_2^{(N)} \geq \dots \geq \lambda_N^{(N)}\right)$

$\Updownarrow$

$x_i^N = \lambda_i^{(N)} + N - i \quad : \quad x_1^N > x_2^N > \dots > x_N^N$

x_i^N encode positions of $\lozenge$ lozenges in a tiling

Under the correspondence GT patterns $\longleftrightarrow$ tilings



Coherent systems in GT

$\updownarrow$

Central measures on paths (GT patterns)

$\updownarrow$

Random tilings satisfying conditional uniformity (or Gibbs) property:

In any subdomain, given the assignment of lozenges along its boundary the conditional law of the tiling inside is uniform.

Classification of coherent systems

$\updownarrow$

Classification of Gibbs measures on tilings

[framework of statistical mechanics]

More about random tilings:

Lectures on random lozenge tilings

Vadim Gorin

http://www.math.wisc.edu/~vadicgor/Random_tilings.pdf

The measures $\mathcal{M}_N^{(\alpha^\pm, \beta^\pm, \gamma^\pm)}$ have a remarkable connection to last passage percolation problem.

For simplicity we take $\alpha_1^+ \geq \alpha_2^+ \geq \dots \geq \alpha_K^+ \geq 0$, other parameters $=0$.

Consider $K \times N$ table filled with independent w_{ij}

Each w_{ij} is a geometric random variable of mean α_i^+

$$q_i = \frac{\alpha_i^+}{1+\alpha_i^+} : \qquad \mathrm{Prob}\,(w_{ij} = k) = \frac{1}{1-q_i} \cdot (q_i)^k, \quad k = 0, 1, 2, \dots$$

[Check: the mean of $\mathrm{Geom}(q) = \frac{q}{1-q}$. $\frac{q_i}{1-q_i} = \frac{\alpha_i^+/(1+\alpha_i^+)}{1-\alpha_i^+/(1+\alpha_i^+)} = \alpha_i^+$]

$K=3$

$N=4$

[One possible path $(1,1) \to (3,4)$]

Definition: $$L(N) = \max_{(i(t), j(t))} \sum_{t=1}^{K+N} w_{i(t), j(t)}$$

where the maximum is taken over all up-right paths $(1,1) \to (K, N)$ on the grid with steps $(1,0)$ or $(0,1)$

Interpretation of $L(N) = \max\limits_{(i(t),j(t))} \sum\limits_{t=1}^{K+N} W_{i(t),j(t)}$:

You have a large group of people who travel from $(1,1)$ to (K,N) by all possible paths. $W_{i,j}$ is the time one needs to spend at (i,j) before proceeding further. Then $L(N)$ is the time when everyone finishes the journey.

Theorem 5: $L(N) \overset{d}{=} \lambda_1^{(N)}$ for $\mathcal{M}_N^{(d_1^+, \dots d_K^+)}$-distributed $\lambda^{(N)}$

first coordinate

Similarly to Week 7, Slides 10-12, the proof goes through the RSK and we do not present it [See MATH 740, Lecture 13, Slide 12]

Benefit 1: Theorem + LLN $\Rightarrow$ $\lim\limits_{N\to\infty} \frac{\lambda_1^{(N)}}{N} = d_1^+$ (K is fixed)

Benefit 2: Theorem + determinantal structure of $\mathcal{M}_N^{(\dots)}$ $\Rightarrow$ $\lambda_1^{(N)}$ as $N, K \to \infty$

Shape Fluctuations and Random Matrices

Kurt Johansson

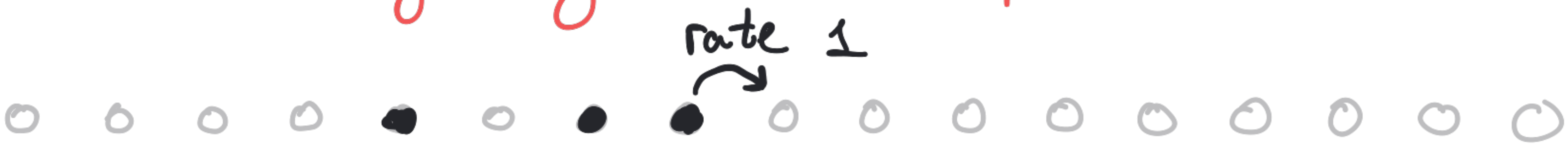
Communications in Mathematical Physics **209**, 437–476(2000) | Cite this article

[see also MATH 740, HW 4 (optional problem there]

The measures $\mathcal{M}_N^{(\alpha^\pm;\beta^\pm;\gamma^\pm)}$ also have a remarkable connection to interacting particle systems.

For simplicity we only deal with the case $\alpha^\pm, \beta^\pm, \gamma^- = 0$; $\gamma^+ > 0$.

Consider Totally Asymmetric Simple Exclusion Process (TASEP):

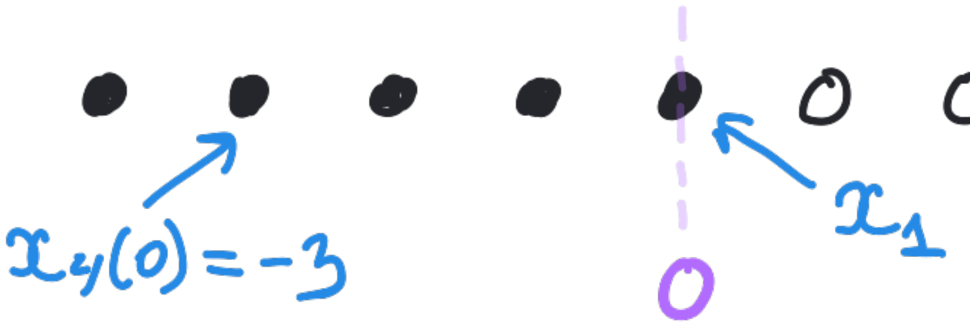


We have particles on the lattice $\mathbb{Z}$. Each particle has an independent exponential clock of rate 1, i.e., the spacings between the times when clock rings are i.i.d. exponential random variables (of density e^{-x}, $x > 0$)

When the clock of a particle at position x rings, it checks whether $x+1$ is empty. If so, then the particle jumps to $x+1$. Otherwise, it is blocked and nothing happens.

TASEP = simplistic model of cars on a 1-lane highway.
Central question of interest: large time behavior?

Theorem 6: Assume that TASEP is started from step initial condition at time 0: $x_i(0) = 1-i$, $i=1,2,\dots$



Then the law at time t of $(x_1(t), x_2(t), x_3(t), \dots)$ is the same as $(\lambda_1^{(1)}, \lambda_2^{(2)}-1, \lambda_3^{(3)}-2, \lambda_4^{(4)}-3, \dots)$

smallest coordinate

Where $\lambda^{(1)} \prec \lambda^{(2)} \prec \lambda^{(3)} \prec \dots$ is a random path in GT distributed according to the extreme central measure with $\alpha_i^{\pm} = \beta_i^{\pm} = 0$, $\gamma = 0$, $\delta^{\pm} = t$

Checks: At $t=0$, $c_k = \delta_{k=0}$ and $\mu_N^{(0)} = \delta_{(\lambda_1=\lambda_2=\dots=\lambda_N=0)}$.
Hence, $\lambda^{(N)} = 0^N$ almost surely and $\lambda_N^{(N)} = 0$, matching $x_N(0) = 1-N$.
For $t>0$, $c_k = e^{-t}\frac{t^k}{k!} \Rightarrow \lambda_1^{(1)}$ has Poisson distribution with mean t — as $x_1(t)$

One approach to the proof of Theorem 6:

Anisotropic Growth of Random Surfaces in 2 + 1 Dimensions

Alexei Borodin & Patrik L. Ferrari

Communications in Mathematical Physics **325**, 603–684(2014) | Cite this article

Multilevel TASEP

There is a stochastic dynamics on GT-patterns, which

- Preserves centrality (central at $t=0$ ⇒ central at all $t>0$)
- Corresponds to deterministic growth of γ^+ on the boundary
- $\left(\lambda_N^{(N)}+1-N\right)_{N=1}^{\infty}$ evolve as TASEP

Benefits: Using determinantal structure of $M_N^{(\gamma^+)}$, one can obtain complete description of $\left(x_i(t)\right)_{i=1}^{\infty}$ for fixed t.

Shape Fluctuations and Random Matrices

Kurt Johansson

Communications in Mathematical Physics **209**, 437–476(2000) | Cite

Advances in Mathematics
Volume 219, Issue 3, 20 October 2008, Pages 894-931

Asymptotics of Plancherel measures for the infinite-dimensional unitary group

Alexei Borodin, Jeffrey Kuan

Dynamics of a Tagged Particle in the Asymmetric Exclusion Process with the Step Initial Condition

T. Imamura & T. Sasamoto

Journal of Statistical Physics **128**, 799–846(2007) | Cite this article

Electronic Journal of Probability

Vol. 13 (2008), Paper no. 50, pages 1380–1418.

Journal URL
http://www.math.washington.edu/~ejpecp/

Large time asymptotics of growth models on space-like paths I: PushASEP

Alexei Borodin, Patrik L. Ferrari