Math 833 Young graph and its deformations Week 7

Last week we found the Martin boundary of the Young graph: it is parameterized by $\alpha_1 \geq \alpha_2 \geq \ldots$, $\beta_1 \geq \beta_2 \geq \ldots$ $\sum(\alpha_i+\beta_i) \leq 1$

Our first task for today: show that these measures

I) Are extreme

II) Satisfy the Law of Large Numbers $\left(\frac{\lambda_i}{n} \to \alpha_i \, ; \; \frac{\lambda_i'}{n} \to \beta_i \right)$

In fact, I and II are equivalent:

- If we know II, then the measures are supported on disjoint sequences of growing Young diagrams. Hence, they are all extreme, like in Week 4, slide 19
- In the opposite direction the limits of $\frac{\lambda_i}{n}$, $\frac{\lambda_i'}{n}$ should exist by Martin boundary computation and be non-random by extremality [details later today]

Proposition 1: All coherent systems of the Martin boundary of the Young graph (found last week) are extreme.

Proof: Given a coherent system $(M_0, M_1, \ldots)$ we construct a linear functional $\varphi^M : \Lambda \to \mathbb{R}$

algebra of symmetric functions

satisfying the condition (H) $\varphi^M(g \cdot p_1) = \varphi^M(g) \quad (\forall g \in \Lambda)$

$x_1 + x_2 + \ldots$

by requiring $\varphi^M(S_\lambda) = \dfrac{M_n(\lambda)}{\dim(\lambda)}$ $\quad \lambda \in \mathbb{Y},\ |\lambda| = n$

Schur functions form a linear basis in Λ

Let us check (H) ["harmonicity condition"]

$$\varphi^M(S_\mu p_1) = \varphi^M\Big(\sum_{\lambda = \mu + \square} S_\lambda\Big) = \sum_{\lambda = \mu + \square} \frac{M_n(\lambda)}{\dim \lambda} = \frac{M_n(\mu)}{\dim(\mu)} = \varphi(S_\mu)$$

Fact 2 on Slide 5, Week 6

coherency relation

Clearly, under correspondence extreme coherent systems are in bijection with extreme functionals $\varphi : \Lambda \to \mathbb{R}$ such that:

- $\varphi(1) = 1$
- $\varphi(g \cdot p_1) = \varphi(g)$ for each $g \in \Lambda$
- $\varphi(S_\lambda) \geq 0$ for each $\lambda \in \mathbb{Y}$

The functionals which correspond to Martin boundary of $\mathbb{Y}$ are very special: they are algebra homomorphism

Indeed, we found last time that $\varphi^{(\alpha,\beta)}(f)$ is obtained by setting $\varphi^{(\alpha,\beta)}(p_k) = \rho_k = \begin{cases} 1, & k=1 \\ \sum_i (\alpha_i)^k + (-1)^{k-1} \beta_i^k, & k>1. \end{cases}$ and extending to the rest by expressing as polynomials in p_k.

In other words, each $\varphi^{(\alpha,\beta)}$ is multiplicative:

For any $f, g \in \Lambda$ $\quad \varphi^{(\alpha,\beta)}(fg) = \varphi^{(\alpha,\beta)}(f) \cdot \varphi^{(\alpha,\beta)}(g)$

Claim: It is impossible to represent a multiplicative functional as a positive linear combination (or integral) of other multiplicative functionals $\varphi^{(\alpha,\beta)}$

This is a particular case of "Ring theorem" of Vershik and Kerov which says that (under certain conditions) extremality $\Leftrightarrow$ multiplicativity

Proof of claim: Assume that $\varphi = \int\limits_{(\alpha,\beta)} \varphi^{(\alpha,\beta)} \, d\nu$, where ν is some probability measure on the set of all (α,β).

Take $f \in \Lambda$ and treat $\varphi^{(\alpha,\beta)}(f)$ as a random variable on probability space of all (α,β) with measure ν

Its expectation $\mathbb{E}\,\varphi^{(\alpha,\beta)}(f) = \int \varphi^{(\alpha,\beta)}(f)\, d\nu = \varphi(f)$

Further, $\mathbb{E}\left[\varphi^{(\alpha,\beta)}(f)\right]^2 \underset{\text{multiplicativity of } \varphi^{(\alpha,\beta)}}{=} \mathbb{E}\left[\varphi^{(\alpha,\beta)}(f^2)\right] = \varphi(f^2) \underset{\text{multiplicativity of } \varphi}{=} (\varphi(f))^2$

Hence, $\mathrm{Variance}\left(\varphi^{(\alpha,\beta)}(f)\right) = 0$ and $\varphi^{(\alpha,\beta)}(f) = \mathrm{const}$ ν-almost surely

We conclude that for all measures in the support of ν and all $f \in \Lambda$ the values of $\varphi^{(\alpha,\beta)}(f)$ are the same. But this is possible only if the support of ν consists of a single point: values of all $\varphi^{(\alpha,\beta)}(p_k) = \sum_i (\alpha_i)^k + (-1)^{k-1}(\beta_i)^k$ uniquely fix (α, β).

$\blacksquare$

The claim implies the proposition: indeed, we know that minimal boundary $\subset$ Martin boundary. But we also know that measures in the Martin boundary can not be represented by integrals of each other. Hence, actually minimal boundary $=$ Martin boundary, i.e. all coherent systems $M^{(\alpha,\beta)}$ are extreme.

$\blacksquare$

Proposition 1 finishes the proof of Theorem 1, week 6

Proof of Week 6, Theorem 2 (LLN for $M^{(\alpha,\beta)}$)

Consider $M^{(\tilde{\alpha},\tilde{\beta})}$ and corresponding central measure on paths in the Young graph. Let $\lambda^{(n)}, n=1,2,\dots$ be a random sequence of Young diagrams distributed by this measure.

Recall that $L_{n\to k}(\lambda,\mu)$ is the law of diagram $\mu\in\mathbb{Y}_k$ with respect to the uniform measure on paths $\varnothing\to\lambda$.

Consider $L_{n\to k}(\lambda^{(n)},\mu)$. This is a random measure on diagrams $\mu\in\mathbb{Y}_k$ [Two levels of randomness!]

Let us look at $\lim\limits_{n\to\infty} L_{n\to k}(\lambda^{(n)},\mu)$. On one hand, by general compactness argument, the limit should exist (at least along a subsequence) and should give a random coherent system.

Since we know that each coherent system is a mixture of extreme systems $M^{(\alpha,\beta)}$, a random coherent system is a random variable taking values in the set of all probability measures on (α,β).

Let us take expectation of this random variable. This is a probability measure on (α,β).

On the other hand, by coherency

[This is the place where we use extremality of $M^{(\hat\alpha,\hat\beta)}$]

$$\mathbb{E}_{\lambda(n)}\, L_{n\to K}(\lambda^{(n)},\mu) = M_K^{(\hat\alpha,\hat\beta)}(\mu)$$

Hence, the expectation is the δ-measure (unit mass) at the point $(\hat\alpha,\hat\beta)$. However, expectation of a random probability measure can be a δ-measure only if this random probability measure is almost surely this probability measure.

Conclusion: $\lim_{n\to\infty} L_{n\to\kappa}(\lambda^{(n)}, \mu) = \mathcal{M}_{\kappa}^{(\mathcal{I},\mathcal{B})}(\mu)$

random measure — deterministic measure

However, by the Martin boundary computation $\mathcal{M}_{\kappa}^{(\mathcal{I},\mathcal{B})}(\mu)$ is a limit if and only if

$$\frac{\lambda_i^{(n)}}{n} \to \mathcal{I}_i \quad \text{and} \quad \frac{(\lambda^{(n)})_i'}{n} \to \mathcal{B}_i . \qquad (*)$$

Hence, we achieve the desired Law of Large Numbers ∎

Remark: Careful analysis of the argument reveals that $(*)$ is both convergence in probability and almost sure convergence by the central measure corresponding to $\mathcal{M}^{(\mathcal{I},\mathcal{B})}$.

There exists an alternative approach, in which one first proves Theorem 2 directly (no use of S^*_λ), and then deduces Theorem 1 as a corollary of LLN and certain stochastic monotonicity of $L_{n\to\kappa}(\lambda,\mu)$ (with respect to λ and μ)

Stochastic Monotonicity in Young Graph and Thoma Theorem
Alexey Bufetov, Vadim Gorin
International Mathematics Research Notices, Volume 2015, Issue 23, 2015, Pages 12920–12940, https://doi.org/10.1093/imrn/rnv085

But how can you prove Theorem 2 directly and avoiding any use of shifted-symmetric functions?
In fact, there are two approaches:

A) If one randomizes n by Poisson distribution, then $M_n^{(\alpha,\beta)}$ turns into a Schur measure of

Infinite wedge and random partitions
A. Okounkov
Selecta Mathematica 7, Article number: 57 (2001) | Cite this article

This implies that random point process $\{\lambda_i - i\}$ can be described by determinants of a matrix of (analyzable) contour integrals.

B) Measures $M_n^{(\alpha,\beta)}$ can be constructed from sequences of i.i.d. random variables by a remarkable bijection.

SIAM. J. on Algebraic and Discrete Methods, 7(1), 116–124. (9 pages)

The Characters of the Infinite Symmetric Group and Probability Properties of the Robinson–Schensted–Knuth Algorithm

Sergei V. Kerov and Anatol M. Vershik

The treatment of α_i, β_i, and γ parameters is slightly different, so let us look into more details on the case $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_K > 0$, $\sum_{i=1}^{K} \alpha_i = 1$, all other parameters $= 0$.

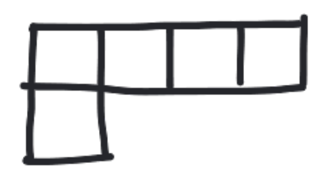
Algorithm: We sample a random word $W = (w_1 w_2 \dots w_n)$ such that each $w_i \in \{1, 2, \dots, K\}$ is an i.i.d. random variable with $\text{Prob}(w_i = j) = \alpha_j$.

A (weakly) increasing subsequence in W is a sequence of indices $i_1 < i_2 \dots < i_m$ such that $w_{i_1} \leq w_{i_2} \leq \dots \leq w_{i_m}$

Example: 13532 has 2 inc. subs. of length 3: $\bar{1}\bar{3}\bar{5}32$ $\quad$ $\bar{1}\bar{3}5\bar{3}2$

Define $\lambda = \lambda(W) \in Y_n$ through

$\lambda_1 + \lambda_2 + \ldots + \lambda_m =$ maximal total length of m disjoint (weakly) increasing subsequences in W

Example: $W = 1\,3\,2\,2\,4 \rightarrow$ 

$\lambda_1 = 4$ $\quad 1\,3\,2\,2\,4$

$\lambda_1 + \lambda_2 = 5$ $\quad 1\,3\,2\,2\,4$

Exercise 1: $W = 3\,5\,1\,1\,2\,2\,5\,4\,2 \longrightarrow\ ?$

Remark: This is a particular case of Robinson-Schensted-Knuth correspondence (RSK): bijection between words of length n in alphabet $\{1, \ldots K\}$ and pairs (semistandard Young tableau, standard Young tableau) of the same shape λ with n boxes and $\leq K$ rows.

Proposition 2: The procedure results in $\mathcal{M}_n^{(\alpha_1,\dots,\alpha_k;0)}$

We do not give a proof. But it is quite straightforward from the properties of RSK. See, e.g., MATH 740, Lec 12-13

For the Plancherel measure $\mathcal{M}_n^{(0,0)}$ $[\gamma=1]$ the construction is similar: take word $w=(w_1 w_2 \dots w_n)$, where w_i are i.i.d. uniform on $[0,1]$ and apply the same procedure.

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swap $\alpha \leftrightarrow \beta$ transposition rows $\leftrightarrow$ columns

$$\mathcal{M}_n^{(\alpha,\beta)}(\lambda) = \mathcal{M}_n^{(\beta,\alpha)}(\lambda')$$

Hence, by transposing at the end, we can also get $\mathcal{M}_n^{(0;\beta_1,\dots\beta_k)}$

Vershik and Kerov also gave a construction for general $\mathcal{M}_n^{(\alpha,\beta)}$, but it is more delicate and we will not present it.

Sketch of the second proof of Theorem 2 (Week 6) for case $(\alpha_1, \dots \alpha_k)$:

Construct $\mathbb{M}_n^{(\alpha_1,\dots\alpha_k;0)}$ by $RSK(W)$.

If n is large, then W has $\alpha_1 n + \bar{o}(n)$ letters 1.
(by standard LLN)

Hence, $\lambda_1 \geq \#\text{letters } 1 = \alpha_1 n + \bar{o}(n)$

they form a subsequence!

For the upper bound, one should notice that adding other letters to the increasing subsequence does not help much — there are not enough of them. Hence, $\lambda_1 \approx \alpha_1 n$.

Similarly, $\lambda_1 + \lambda_2 \approx \#\text{letters } 1 \text{ and } 2 \approx \alpha_1 n + \alpha_2 n$

Continuing, we get $\lambda_i \approx \alpha_i n$ for all $i = 1, \dots k$.

For further details and CLT for fluctuations of λ_i by the same method see →

A central limit theorem for extremal characters of the infinite symmetric group

Alexey I. Bufetov

Functional Analysis and Its Applications 46, 83–93(2012) | Cite this article

The Young graph has an important deformation, which depends on a pair of real parameters (q,t)

The deformation uses as a starting point a face of the boundary of Young graph, which we found in the proof of Proposition 1. Coherent systems $M^{(\alpha,\beta)}$ give an answer to the following problem:

- What are all possible Schur-positive algebra homomorphisms $\varphi: \Lambda \to \mathbb{R}$, such that $\varphi(s_\lambda) \geq 0$ for each $\lambda \in \mathbb{Y}$?

The answer is given by $\varphi(p_k) = \begin{cases} \sum \alpha_i + \sum \beta_i + \gamma, & k=1, \\ \sum (\alpha_i)^k + (-1)^{k-1} \sum (\beta_i)^k, & k>1. \end{cases}$ $\quad \begin{array}{l} \alpha_i \geq 0 \\ \beta_i \geq 0 \\ \gamma \geq 0 \end{array}$

[We had a normalization $\varphi(p_1) = \sum \alpha_i + \sum \beta_i + \gamma = 1$, but this is not important: transformation $\varphi \to \varphi_r$: $\varphi_r(f) = r^m \varphi(f)$ for homogeneous f of degree m preserves positivity and allows to fix $\varphi(p_1)$ however we want]

In this formulation we can replace Schur functions s_λ by any other homogeneous linear basis in Λ.

Kerov proposed to study a version for Macdonald polynomials $P_\lambda(\cdot;q,t)$

[You don't have to know who they are. But if you want, you can start from MATH 740, Lecture 15]

He conjectured that for $|q|,|t|<1$ Macdonald-positive homomorphisms $\varphi:\Lambda\to\mathbb{R}$: $\varphi(P_\lambda)\geq 0$ admit a description:

$$\varphi(p_k)=\begin{cases}\sum\alpha_i+\sum\beta_i+\gamma, & k=1\\ \sum(\alpha_i)^k+(-1)^{k-1}\dfrac{1-q^k}{1-t^k}\left(\dfrac{1-t}{1-q}\right)^k\sum(\beta_i)^k, & k>1.\end{cases}$$

This result was proven 25 years later

Macdonald-positive specializations of the algebra of symmetric functions: Proof of the Kerov conjecture

Konstantin Matveev

Annals of Mathematics
Vol. 189, No. 1 (January 2019), pp. 277-316 (40 pages)

Remark: The branching graph corresponding to this is (q,t)-weighted Young graph, in which each edge comes with a weight and probability of a path is proportional to the product of weights of egges. Matveev: minimal boundary. LLN? Still open!

Macdonald polynomials are complicated, but they have many much simpler degenerations at special values of (q,t)

$q=0$, $t=1$ gives monomial symmetric functions m_λ

$$m_\lambda = \sum_{\substack{(a_1,a_2,\dots) \text{ is a permutation} \\ \text{of } (\lambda_1,\lambda_2,\dots)}} x_1^{a_1} \cdot x_2^{a_2} \cdot x_3^{a_3} \cdot \dots \in \Lambda$$

$[\lambda \in \mathbb{Y}]$

Examples: $m_{(2)} = \sum_i (x_i)^2$ $\qquad m_{(2,1)} = \sum_{i \neq j} (x_i)^2 x_j$

Theorem A: All algebra homomorphisms $\varphi: \Lambda \to \mathbb{R}$ satisfying $\varphi(m_\lambda) \geq 0$ for all $\lambda \in \mathbb{Y}$ are of the form:

$$\varphi(p_k) = \begin{cases} \sum_i d_i + \gamma, & k=1, \\ \sum_i (d_i)^k, & k>1. \end{cases}$$

$d_1 \geq d_2 \geq \dots \geq 0$, $\sum d_i < \infty$, $\gamma \geq 0$.

Note that β-parameters disappear in this case!

A statement equivalent to Theorem A was known already to

John Frank Charles Kingman

Quick Info

Born
28 August 1939
Beckenham, Kent, England

Journal of the London Mathematical Society
LONDON MATHEMATICAL SOCIETY EST. 1865

Volume s2-18, Issue 2
October 1978
Pages 374-380

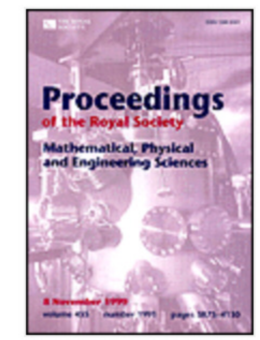
Notes and Papers

The Representation of Partition Structures

Random Partitions in Population Genetics

J. F. C. Kingman

Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences
Vol. 361, No. 1704 (May 3, 1978), pp. 1-20 (20 pages)



He is a probabilist and statistician, but not an algebraist.
He was solving a different problem!

Definition: Partition structure is a random partitioning of $\mathbb{Z}_{>0} = \{1,2,3,\dots\}$ into disjoint subsets, such that the distribution is invariant under permutations (= action of $S(\infty)$)

Examples:

- Partition into a single set $\mathbb{Z}_{>0} = \mathbb{Z}_{>0}$
- Partition into singletons $\mathbb{Z}_{>0} = \bigsqcup_{i>0} \{i\}$

These are the only deterministic partition structures

Theorem B: Extreme partition structures are parameterized by $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$, $\gamma \geq 0$: $\sum_i \alpha_i + \gamma = 1$. Explicitly the corresponding partition is constructed by a stochastic algorithm:

Let ξ be a random variable taking values in $\{\mathcal{D}, 1, 2, 3, \ldots\}$
"dust"

$$\mathrm{Prob}(\xi = \mathcal{D}) = \gamma \qquad \mathrm{Prob}(\xi = i \geq 1) = \alpha_i$$

Sample a sequence of i.i.d. random variables $\xi_1, \xi_2, \ldots$ of distribution ξ each. For each $n = 1, 2, \ldots$ we set:

- If $\xi_n = i$, then n belongs to the i-th set
- If $\xi_n = \mathcal{D}$, then n forms its own set – singleton $\{n\}$

Remark: While in this construction the sets happened to be ordered, but we discard this order eventually: for instance, there is only one (not two!) partitions $\mathbb{Z}_{>0} = \mathrm{odd} \sqcup \mathrm{even}$

We will not prove Theorems A or B. [They are not hard given the tools that we already have, but will distract us.]
Instead we show how they connect to each other and to coherency.

Question: What is a partition structure on $\{1, 2, .. n\}$?

[E.g. $\{1,3\} \sqcup \{2,4\}$ is a set partition of $\{1,2,3,4\}$]

Note that any two set partitions of $\{1, .., n\}$ with the same sizes of sets (given by partitition $\lambda \in \mathbb{Y}_n$) can be obtained one from another by a permutation of elements. Hence, their probabilities should be the same. Therefore, we have proven:

Proposition 3: Partition structures on $\{1, .. n\}$ (= random $S(n)$-invar. set partitions) are in bijection with probability measures on $\mathbb{Y}_n$. The latter compute the distribution of the sizes of blocks in set partition. Conditionally on these sizes, the distribution is necessarily uniform.

Exercise 2: Compute $\dim^{K}(\lambda) = \#\{\text{set partitions with blocks } \lambda\}$

["K" stays for Kingman]

Example: There are 3 set partitions of $\{1,2,3,4\}$ into two sets of two elements each: $\{1,2\} \sqcup \{3,4\}$; $\{1,3\} \sqcup \{2,4\}$; $\{1,4\} \sqcup \{2,3\}$

Hence, $\dim^{K}(2,2) = 3$.

Conclusion: Partition structure on $\mathbb{Z}_{>0}$ can be identified with a sequence of probability measures M_n on $\mathbb{Y}_n$, corresponding to restrictions on $\{1,2,\dots n\}$

How are they related to each other?

Is it a coherent system on the Young graph?

No! Similar combinatorially, but cotransition probabilities are different

Proposition 4: The coherency relation takes the form

$$M_n(\mu) = \sum_{\lambda = \mu + \square} K(\mu, \square) \frac{\dim^{\kappa}(\mu)}{\dim^{\kappa}(\lambda)} M_n(\lambda) \quad (*)$$

where $K(\mu, (i,j)) = \#\{a : \mu_a = \mu_i\}$

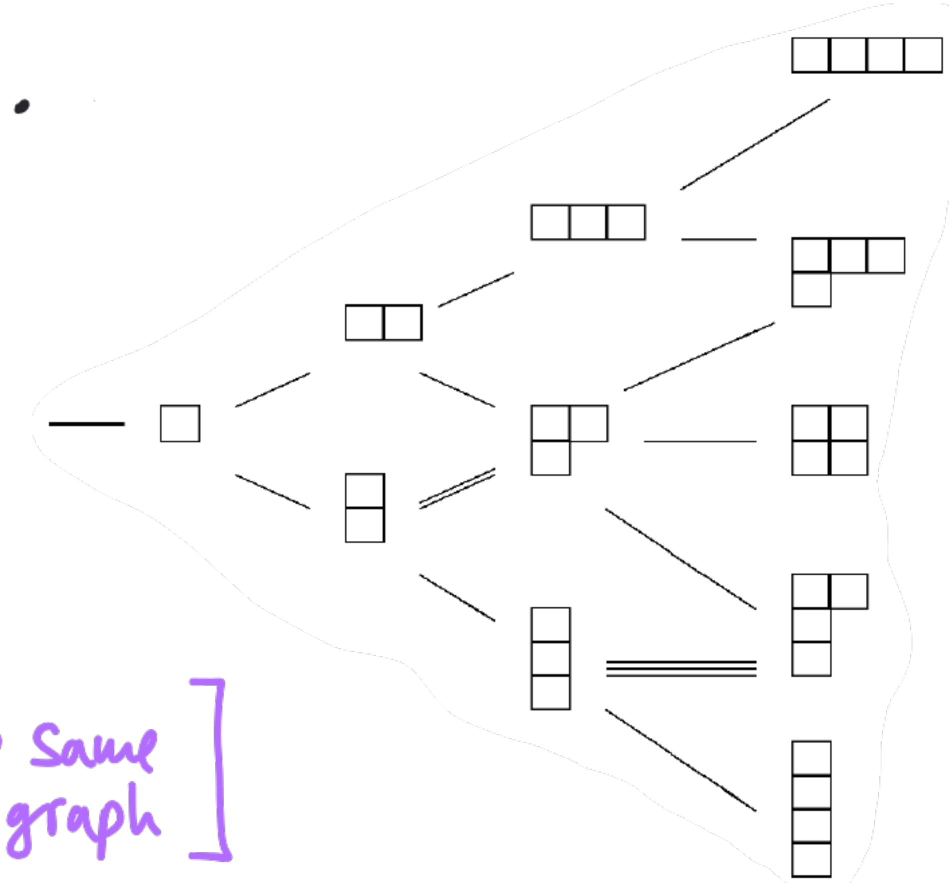
row of $\square$ column of $\square$

Proof: We compute cotransitional probabilities and show that they equal (*). We need to compute the conditional probability of blocks μ in set partition of $\{1, .., n\}$ given blocks λ in $\{1, .., n+1\}$.

Fix λ. There are $\dim^{\kappa}(\lambda)$ set partitions with blocks λ. Out of them $\dim^{\kappa}(\mu) \cdot K(\mu, (i,j))$ have blocks μ when restricted to $\{1, .. n\}$: Indeed, we first fix set partition with blocks μ in $\dim^{\kappa}(\mu)$ ways and then choose one of $K(\mu, (i,j))$ sets of size μ_i and add $\{n+1\}$ to it. ∎

Ratio gives (*).

This is the Kingman graph.



The first 4 levels taken from the Kerov's book, who calls it „Bell's scheme"

[several closely related graphs with the same boundary can be also called Kingman graph]

Combinatorially like the Young graph, but with weigths/multiplicities of edges. $\dim^K(\lambda)$ is the dimension in this graph.

Conclusion: we identified Theorem B with computation of the boundary of the Kingman graph.

Theorem A can be also identified with it by expanding $m_\mu \cdot p_1 = \sum_{\lambda=\mu+\square} (?)\, m_\lambda$ and another instance of Ring theorem (multiplicativity $\Longleftrightarrow$ extremality)