Math 833 Boundary of the Young graph Week 6

Theorem 1: The extreme coherent systems on $\mathbb{Y}$ are parameterized by pairs of real sequences:

$\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$, such that $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1$. With the notation $\gamma = 1 - \sum_{i=1}^{\infty} (\alpha_i + \beta_i)$ the corresponding coherent system is:

$$M_n^{(\alpha,\beta)}(\lambda) = \dim \lambda \cdot \det \left[C_{\lambda_i - i + j} \right]_{i,j=1}^{m} \qquad (*)$$

[m can be any number $\geq$ number of rows in λ]

Where $(**)$ $\sum_{k \in \mathbb{Z}} C_k z^k = e^{\gamma z} \cdot \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}$, in particular $C_k = 0$ for $k < 0$

First proofs of equivalent statements: Schoenberg, Edrei, Thoma

Our today's approach: Vershik-Kerov. Kerov-Okounkov-Olshanski

The parameters α_i, β_i have a clean probabilistic meaning, discovered by Vershik and Kerov

Theorem 2: In the notations of the last slide let $\lambda^{(n)}$ be $M_n^{(\alpha,\beta)}$-distributed random Young diagram with n boxes. Then for each $i = 1, 2, \dots$ we have:

[limits in probability]

$$\lim_{n\to\infty} \frac{\lambda_i^{(n)}}{n} = \alpha_i \qquad\qquad \lim_{n\to\infty} \frac{(\lambda^{(n)})'_i}{n} = \beta_i$$

length of the i-th row length of the i-th column

Aim for today: Prove Theorems 1 and 2 | This is a very algebraic week!

Plan of the proof:

1) We identify the Martin boundary by finding

$$(***) \quad \lim_{|\lambda|\to\infty} \frac{\dim(\mu)\cdot\dim(\lambda/\mu)}{\dim(\lambda)} \quad \text{with } \mu \text{ fixed}$$

#paths $\varnothing \to \mu$ #paths $\mu \to \lambda$

#paths $\varnothing \to \lambda$

Based on explitit formula through shifted Schur functions

Алгебра и анализ, 1997, том 9, выпуск 2, страницы 73–146
(Mi aa762)

Эта публикация цитируется в 114 научных статьях

Статьи

Сдвинутые функции Шура

А. Окуньков[a], Г. Ольшанский[b]

arXiv.org > q-alg > arXiv:q-alg/9605042

Quantum Algebra and Topology

[Submitted on 28 May 1996]

Shifted Schur Functions

Andrei Okounkov, Grigori Olshanski

We will see that (***) implies $\frac{\lambda_i}{n} \to \alpha_i$, $\frac{\lambda_i'}{n} \to \beta_i$

2) Match the limit with Theorem 1 using (homogeneous) Schur functions

3) Prove extremality using the identification of coherent systems with Schur-positive functionals on symmetric functions.

We use facts from theory of symmetric functions as a black box. If you want details in this direction, review MATH 740 (Fall 2020, Lec. 1-9 and 23-24)

MATH740: Enumerative Combinatorics/Symmetric Functions (001) FA20
canvas.wisc.edu/courses/219345

List of facts:

Λ = algebra of symmetric functions = polynomial expressions in $x_1, x_2, \ldots$ of bounded degree and invariant under permutations of variables (action of $S(\infty)$)

Examples of elements:

Power sums $p_k = \sum_i (x_i)^k = (x_1)^k + (x_2)^k + \ldots$

Elementary symmetric functions $e_k = \sum_{i_1 < \ldots < i_k} x_{i_1} x_{i_2} \cdot \ldots \cdot x_{i_k}$

Complete homogeneous symmetric functions $h_k = \sum_{i_1 \le \ldots \le i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$

Fact 1: $\{p_k\}_{k \geq 1}$ or $\{e_k\}_{k \geq 1}$ or $\{h_k\}_{k \geq 1}$ are algebraically independent generators of Λ

Projections $\rho_N : \Lambda \to \Lambda_N$ ← symmetric polynomials in $\{x_1, \ldots x_N\}$

set $x_{N+1} = x_{N+2} = \ldots = 0.$

function $f \in \Lambda$ $\longleftrightarrow$ sequence $f_N = \rho_N(f)$, $N = 1, 2, \ldots$

(homogeneous) Schur symmetric functions

$$s_\lambda(x_1, \ldots x_N) = \frac{\det\left[x_i^{\lambda_j + N - j}\right]_{i,j=1}^{N}}{\prod_{1 \le i < j \le N} (x_i - x_j)}$$

arbitrary partition with $\le N$ rows

skew-symmetric polynomial

$s_\lambda \in \Lambda$ is an infinite-variable version. They form a basis of Λ

Fact 2: $s_\mu p_1 = \sum_{\lambda = \mu + \square} s_\lambda$ [Proof is easy from definition]

Connects Schur functions to the Young graph!

Fact 3: $s_\lambda = \det\left[h_{\lambda_i - i + j}\right]_{i,j=1}^{m}$ [$m \geq$ number of rows in λ, $h_0 = 1$, $h_{-k} = 0$, $k > 0$. Proof takes efforts] [Compare slide 1 (*)]

Λ^* = algebra of shifted-symmetric functions = polynomial expressions in $x_1, x_2, ..$ of bounded degree symmetric in shifted variables $(x_i - i)$, $i = 1, 2, ..$

Examples of elements:

$$p_k^* = \sum_{i \geq 1} \left[(x_i - i)^k - (-i)^k \right]$$

$$e_k^* = \sum_{i_1 < \ldots < i_k} (x_{i_1} + k - 1) \cdot \ldots \cdot (x_{i_{k-1}} + 1)\, x_{i_k}$$

$$h_k^* = \sum_{i_1 \leq \ldots \leq i_k} (x_{i_1} - k + 1) \cdot \ldots \cdot (x_{i_{k-1}} - 1)\, x_{i_k}$$

Highest degree homogeneous part: p_k, e_k, h_k

Fact 4: $\{p_k^*\}$ or $\{e_k^*\}$ or $\{h_k^*\}$ are algebraically independent generators of Λ^* [Can be deduced from fact 1]

Projections $p_N^* : \Lambda^* \to \Lambda_N^*$ [polynomials symmetric in $\{(x_1 - 1), \ldots (x_N - N)\}$]

set $x_{N+1} = x_{N+2} = .. = 0$

function $f \in \Lambda^*$ $\longleftrightarrow$ sequence $f_N = p_N^*(f)$

Exercise 1: Compute $p_N^*(e_2^*)$ and $p_N^*(h_2^*)$ and show that they are indeed symmetric in variables $\{(x_i - i)\}_{i=1}^N$.

Definition (Shifted Schur functions)

$$S_\mu^*(x_1, \ldots x_N) = \frac{\det\left[(x_i + N - i \downarrow \mu_j + N - j)\right]_{i,j=1}^N}{\underbrace{\det\left[(x_i + N - i \downarrow N - j)\right]_{i,j=1}^N}_{= \prod_{i<j}(x_i - i - (x_j - j))}}$$

skew-symmetric in $x_i - i$

$$(x \downarrow n) = x(x-1)\ldots(x-n+1) \qquad (x \downarrow 0) = 1$$

The infinite-variable version is S_μ^*, they form a basis of Λ^*

The highest degree homogeneous component of S_μ^* is S_μ [directly comparing the definitions]

Fact 5 $\quad S^*_\mu(p_1 - |\mu|) = \sum_{\lambda = \mu + \square} S^*_\lambda$ [In highest degree this is Fact 2]

Fact 6 Interpolation property

number of boxes $\lambda_1 + \dots + \lambda_N$

$S^*_\mu(\lambda_1, \lambda_2, \dots \lambda_N) = 0$ for each $\lambda \in \mathbb{Y}$ such that $|\lambda| \le |\mu|$, $\lambda \ne \mu$

[both μ and λ are assumed to have $\le N$ rows]

Exercise 2: Check Fact 6 for $N=1$ and one-row Young diagrams μ and λ

Theorem 3: Take two Young diagrams λ and μ

#paths $\mu \to \lambda$ in $\mathbb{Y}$ $\qquad$ $s^*_\mu(\lambda_1, \lambda_2, \ldots)$

$$\frac{\dim \lambda/\mu}{\dim \lambda} = \frac{s^*_\mu(\lambda)}{(|\lambda| \downarrow |\mu|)}$$

#paths $\varnothing \to \lambda$ in $\mathbb{Y}$ $\qquad$ as in Slide 7

Proof: Suppose $|\lambda| = n$ and $|\mu| = m$. If $m > n$, then both sides vanish. Otherwise, iterating Fact 5 we get

$$s^*_\mu(p_1 - m) \ldots (p_1 - n + 1) = \sum_{\nu : |\nu| = n} \dim \nu/\mu \ s^*_\nu$$

Evaluate both sides on $\lambda = (\lambda_1, \lambda_2, \ldots)$ using Fact 6. We get

$$s^*_\mu(\lambda) \cdot (n-m)! = \dim \lambda/\mu \cdot s^*_\lambda(\lambda)$$

Choose in the last formula $\mu = \varnothing$ to get $s^*_\lambda(\lambda) = \dfrac{n!}{\dim \lambda}$.

Hence, for general μ: $\quad s^*_\mu(\lambda) \cdot \dfrac{(n-m)!}{n!} = \dfrac{\dim \lambda/\mu}{\dim \lambda}$ $\quad \blacksquare$

Proposition 1: Suppose that a sequence $\lambda^{(n)} \in \mathbb{Y}_n$, $n=1,2,\ldots$ is such that for each $\mu \in \mathbb{Y}$, the sequence $(\dim \lambda^{(n)}/\mu)/(\dim \lambda^{(n)})$ converges. Then there exist $\alpha_1 \geq \alpha_2 \geq \ldots$, $\beta_1 \geq \beta_2 \geq \ldots$, $\sum(\alpha_i+\beta_i) \leq 1$, such that $\frac{\lambda_i^{(n)}}{n} \to \alpha_i$, $\frac{(\lambda^{(n)})'_i}{n} \to \beta_i$ for each $i=1,2,\ldots$

Moreover, then $\lim\limits_{n\to\infty} \frac{\dim \lambda^{(n)}/\mu}{\dim \lambda^{(n)}} = \det\left[c_{\lambda_i - i + j}\right]_{i,j=1}^{m}$, with c_k as in (*), (**) on Slide 1

Proof: By Theorem 3 we need to study the limit

$$\frac{s^*_\mu(\lambda^{(n)})}{n(n-1)\ldots(n-|\mu|+1)} \overset{\text{as } n\to\infty}{\sim} n^{-|\mu|} s^*_\mu(\lambda^{(n)})$$

Since s^*_μ has degree $|\mu|$ and they form a basis of Λ^* we get a statement:

$$\lim_{n\to\infty} \frac{\dim(\lambda^{(n)}/\mu)}{\dim \lambda^{(n)}} \quad \Longleftrightarrow \quad \lim_{n\to\infty} \frac{f(\lambda^{(n)})}{n^{\deg(f)}}$$

exists for all $\mu \in \mathbb{Y}$ $\qquad$ exists for all $f \in \Lambda^*$

Since Λ^* is an algebra generated by p_k^*, it suffices to deal with the latter. Hence, we are led to study: for which sequences $\lambda^{(n)}$ the limit

$$\lim_{n\to\infty} \frac{p_k^*(\lambda^{(n)})}{n^k}$$ exists for all $k=1,2,\dots$?

[Compare with conclusion on Week 2, Slide 8!]

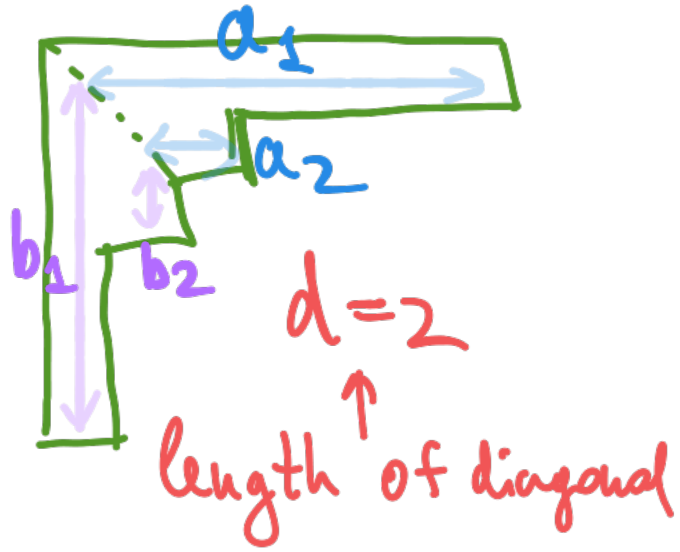
It is convenient to argue in terms of generating functions:

$$\sum_{k\geq 1} p_k^*(\lambda^{(n)}) \frac{z^k}{n^k} \cdot \frac{1}{k} = \sum_{k\geq 1}\left(\sum_{i\geq 1}\left[(\lambda_i^{(n)}-i)\frac{z}{n}\right]^k \cdot \frac{1}{k} - \sum_{i\geq 1}\left[(-i)\frac{z}{n}\right]^k \frac{1}{k}\right) \overset{\text{summing over } k \text{ first}}{=}$$

$$= \sum_{i\geq 1}\left[-\ln\left(1-\frac{z}{n}(\lambda_i^{(n)}-i)\right) + \ln\left(1-\frac{z}{n}(-i)\right)\right] = \ln \prod_{i\geq 1} \frac{1+z\frac{i}{n}}{1-z\frac{\lambda_i^{(n)}-i}{n}}$$

Let us simplify the infinite product:

Lemma: For each $\lambda \in \mathbb{Y}$:
$$\prod_{i \geq 1} \frac{v - \frac{1}{2} + i}{v - \frac{1}{2} - \lambda_i + i} = \prod_{i=1}^{d} \frac{v + b_i}{v - a_i}$$



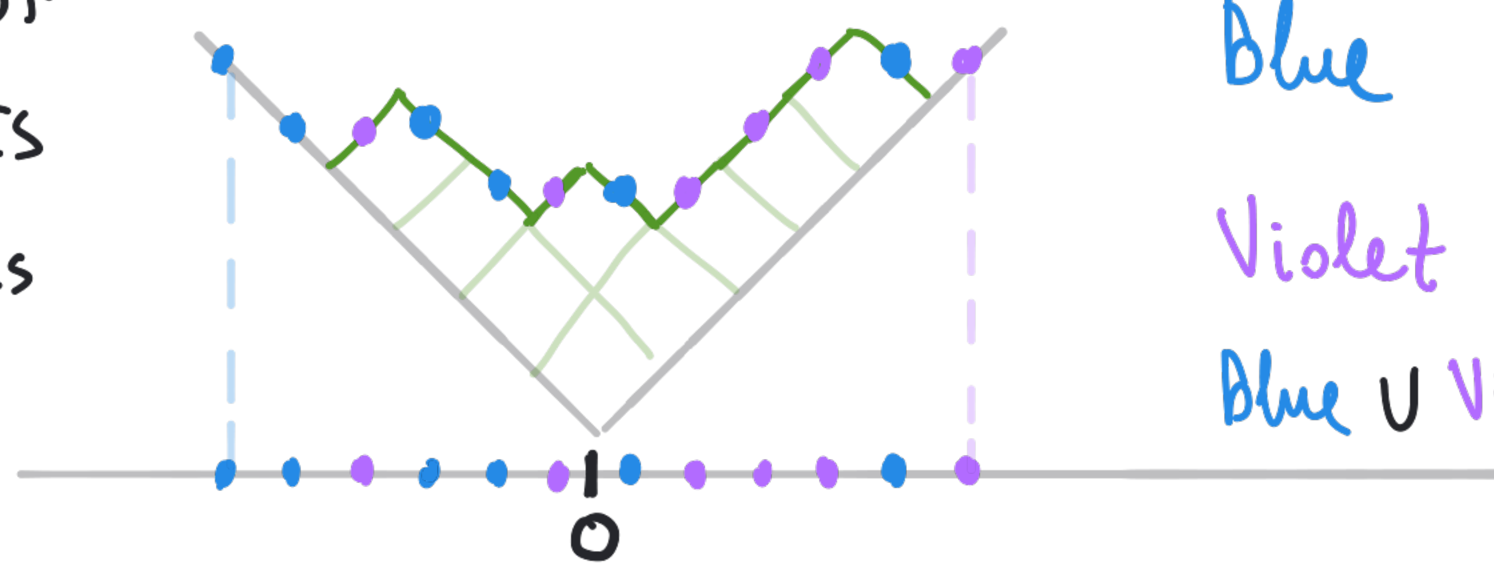
$(a_i, b_i)_{i=1}^{d}$ – modified Frobenius coordinates

$a_i = \lambda_i - i + \frac{1}{2}$ ← rows

$b_i = \lambda_i' - i + \frac{1}{2}$ ← columns

Proof of Lemma: Use Russian style of drawing.

Project midpoints of / and \ segments on the horizontal axis



Blue $\{\lambda_i - i + \frac{1}{2}\}_{i=1}^{\infty}$

Violet $\{-\lambda_i' + i - \frac{1}{2}\}_{i=1}^{\infty}$

Blue ∪ Violet $= \mathbb{Z} + \frac{1}{2}$

Blue $- \{\frac{1}{2} - i\}_{i \geq 1}$ = $\underbrace{\text{Positive blue}}_{\{a_i\}_{i=1}^{d}}$ − $\underbrace{\text{Negative violet}}_{\{b_i\}_{i=1}^{d}}$ $\blacksquare$

Using Lemma we rewrite

$$\sum_{k\geq 1} p_k^*(\lambda^{(n)})\left(\frac{z}{n}\right)^k \cdot \frac{1}{k} = \ln\left(\prod_{i\geq 1}\frac{v+i}{v-\lambda_i^{(n)}+i}\right)_{v=\frac{n}{z}} =$$

$$= \ln\left(\prod_{i=1}^{d^{(n)}}\frac{v+\frac{1}{2}+b_i^{(n)}}{v+\frac{1}{2}-a_i^{(n)}}\right)_{v=\frac{n}{z}} = \ln\prod_{i=1}^{d^{(n)}}\frac{1+z\,\frac{b_i^{(n)}+\frac{1}{2}}{n}}{1-z\,\frac{a_i^{(n)}-\frac{1}{2}}{n}} =$$

$$= \sum_{k\geq 1}\frac{z^k}{k}\sum_{i=1}^{d^{(n)}}\left[\left(\frac{a_i^{(n)}-\frac{1}{2}}{n}\right)^k + (-1)^{k-1}\left(\frac{b_i^{(n)}+\frac{1}{2}}{n}\right)^k\right]$$

Claim: Each coefficient converges if and only if

$$\lim_{n\to\infty}\frac{a_i^{(n)}}{n} = \alpha_i, \quad \lim_{n\to\infty}\frac{b_i^{(n)}}{n} = \beta_i, \quad \alpha_1\geq\alpha_2\geq\ldots, \quad \beta_1\geq\beta_2\geq\ldots, \quad \sum(\alpha_i+\beta_i)\leq 1.$$

The limit is then $\displaystyle\lim_{n\to\infty}\frac{p_k^*(\lambda^{(n)})}{n^k} = \begin{cases} 1, & k=1, \\ \sum_i \alpha_i^k + (-1)^{k-1}\sum_i \beta_i^k. \end{cases}$

Proof of the claim: Similar argument to Week 2, Slide 9, Lemma 1

1) Suppose that $\lim_{n\to\infty} \frac{a_i^{(n)}}{n} = \alpha_i$; $\lim_{n\to\infty} \frac{b_i^{(n)}}{n} = \beta_i$. Then

A) $\frac{1}{n} p_1^*(\lambda^{(n)}) = \frac{1}{n} \sum_i \lambda_i^{(n)} = 1 \xrightarrow[n\to\infty]{} 1$

B) For $k \geq 2$: $\sum_{i=1}^{d} \left(\frac{a_i^{(n)} - \frac{1}{2}}{n} \right)^k + (-1)^{k-1} \sum_{i=1}^{d} \left(\frac{b_i^{(n)} + \frac{1}{2}}{n} \right)^k$

[using $k=1$ case for tail bound] $\Big\downarrow n\to\infty$

$$\sum \alpha_i^k + (-1)^{k-1} \sum \beta_i^k$$

2) Suppose that all $\frac{1}{n^k} p_k^*(\lambda^{(n)}) \to \rho_k$

Then passing to a subsequence we can guarantee convergence of $\frac{a_i^{(n)}}{n}$, $\frac{b_i^{(n)}}{n}$ to some limits. But limits are uniquely fixed by ρ_k [they are poles of z-derivative of $\ln\left(e^{\gamma z} \prod_{i\geq 1} \frac{1+\beta_i z}{1-\alpha_i z}\right)$, to which the generating function converges] ☑

Conclusion: $\dfrac{\dim \lambda^{(n)}/\mu}{\dim \lambda^{(n)}}$ converges as $n\to\infty$ for all μ

if and only if

$$\lim_{n\to\infty} \frac{b_i^{(n)}+\frac{1}{2}}{n} = \beta_i\,; \qquad \lim_{n\to\infty} \frac{a_i^{(n)}-\frac{1}{2}}{n} = \alpha_i$$

or equivalently $\displaystyle \lim_{n\to\infty} \frac{\lambda_i^{(n)}}{n}$ $\qquad$ $\displaystyle \lim_{n\to\infty} \frac{(\lambda^{(n)})'_i}{n}$

Since $a_1^{(n)} > a_2^{(n)} > \ldots \Rightarrow \alpha_1 \geq \alpha_2 \geq \ldots$. Since $b_1^{(n)} > b_2^{(n)} > \ldots \Rightarrow \beta_1 \geq \beta_2 \geq \ldots$

Since $\sum (a_i^{(n)} + b_i^{(n)}) = n \Rightarrow \sum(\alpha_i+\beta_i) \leq 1$ [but do not have to be $=1$]

Further, in this situation

$$n^{-k} p_k^{\#}(\lambda^{(n)}) \longrightarrow p_k = \begin{cases} 1, & k=1, \\ \sum_i (\alpha_i)^k + (-1)^{k-1} \sum_i (\beta_i)^k, & k>1. \end{cases}$$

And we also have $\displaystyle \sum_{k\geq 1} p_k \frac{z^k}{k} = \ln \left[e^{\gamma z} \prod_{i\geq 1} \frac{1+\beta_i z}{1-\alpha_i z} \right]$

Remains to find the formula for $\lim\limits_{n\to\infty} s^*_\mu(\lambda^{(n)}) \cdot n^{-|\mu|}$

Express $s^*_\mu = g_\mu(p_1^*, p_2^*, \ldots)$ $\quad (\Delta)$

g_μ finite degree polynomial

this is possible because p_k^* are generators of Λ^*

Under the agreement $\deg p_k^* = k$, let $\hat{g}_\mu$ be the highest degree part of g_μ. Then

$$S_\mu = \hat{g}_\mu(p_1, p_2, \ldots)$$ is the highest degree of (Δ)

On the other hand, $\lim\limits_{n\to\infty} s^*_\mu(\lambda^{(n)})\, n^{-|\mu|} =$

$$= \lim_{n\to\infty} \hat{g}_\mu\left(\frac{p_1^*}{n}, \frac{p_2^*}{n^2}, \ldots\right) + \underbrace{\lim_{n\to\infty} n^{-|\mu|}(g-\hat{g})(p_1^*, p_2^*, \ldots)}_{\text{vanishes, because } g-\hat{g} \text{ has degree } <|\mu|}$$

Therefore,

$$\lim_{n\to\infty} \frac{s^*_\mu(\lambda^{(n)})}{n^{|\mu|}} = \hat{g}_\mu(p_1, p_2, \ldots) \quad \text{and} \quad s_\mu = \hat{g}_\mu(p_1, p_2, \ldots)$$

How does it match the formula of Proposition 1?

The above formula equivalently says that

$$\lim_{n\to\infty} \frac{s^*_\mu(\lambda^{(n)})}{n^{|\mu|}} = \rho(s_\mu),$$ where $\rho : \Lambda \to \mathbb{R}$ is the algebra homomorphism such that $\rho(p_k) = p_k$.

On the other hand, using Fact 3

$$\rho(s_\mu) = \rho\left(\det[h_{\lambda_i - i + j}]\right) = \det[\rho(h_{\lambda_i - i + j})]$$

Is $\rho(h_k) = c_k$? Well, yes! Because we can apply ρ to

using def. of h_k

$$\sum_k h_k z^k = \prod_i (1 - x_i z)^{-1} = \exp\left(\sum \ln(1 - x_i z)\right) = \exp\left(\sum_{k\ge1} p_k \frac{z^k}{k}\right)$$

Hence, $\sum_{k\geq 0} \rho(h_k) z^k = \exp\left(\sum_{k\geq 1} p_k \frac{z^k}{k}\right) = e^{\gamma z} \prod_{i\geq 1} \frac{1+\beta_i z}{1-\alpha_i z}$ matching (**) from slide 1.

Proposition 1 is proven.

Remark: The formula for p_k which we obtained on slide 15 matches Theorem 5, Week 5, slide 17

In order to show that this is the same formula, one needs to know how the characters of $S(n)$ are related to symmetric functions [MATH 740, Lecture 7] and use correspondence between coherent systems and characters of Proposition, Week 5, Slide 18

Corollary: The Martin boundary of the Young graph is given by the measures of Theorem 1 on Slide 1.

Proof: In the notations of Week 4

$$L_{n\to k}(\lambda^{(n)}, \mu) = \dim \mu \cdot \frac{\dim \lambda^{(n)}/\mu}{\dim \lambda^{(n)}}$$

[Counting paths $\varnothing \to \mu \to \lambda^{(n)}$ out of all paths $\varnothing \to \lambda^{(n)}$]

and it remains to use Proposition 1. $\blacksquare$

Next week: we show extremality of the measures of Theorem 1 and Law of Large Numbers of Theorem 2

[they will turn out to be essentially equivalent]