Math 833 — The Young graph — Week 5

One of the central successes of Integrable Probability is the asymptotic analysis of random Young diagrams. Today we start discussing the branching graph responsible for that.

Definition: A partition of $n = 1, 2, 3, \ldots$ is a way to write n as a sum of positive integers. For example:

- There is one partition of 1: $1 = 1$.
- There are two partitions of 2: $2 = 2$, $2 = 1+1$.
- There are three partitions of 3: $3 = 3$, $3 = 2+1$, $3 = 1+1+1$.
- There are five partitions of 4: $4 = 4$, $4 = 3+1$, $4 = 2+2$, $4 = 2+1+1$, $4 = 1+1+1+1$.

Notation: Y_n = set of all partitions of n

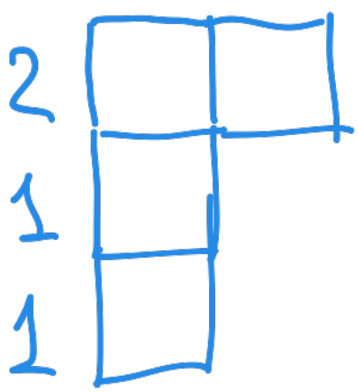
[Agreement $Y_0 = \{\varnothing\}$ - set with one element]

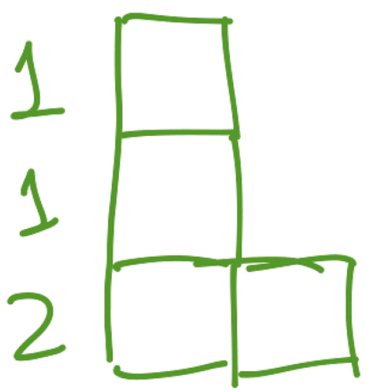
$$Y = \bigsqcup_{n=0}^{\infty} Y_n$$

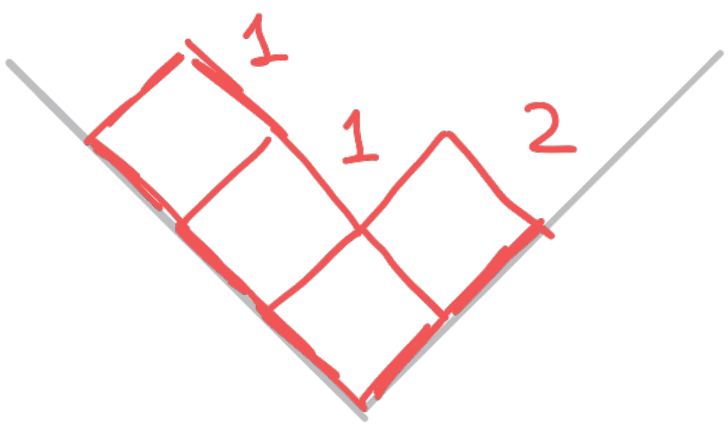
We draw partitions as Young diagrams

[Collections of boxes forming rows representing summands]

$$4 = 2 + 1 + 1$$







English style | French style | Russian style

Definition: The Young graph is a graded graph with vertices $Y = \bigcup_{n \geq 0} Y_n$ and edges connecting diagrams which differ by addition of a single box

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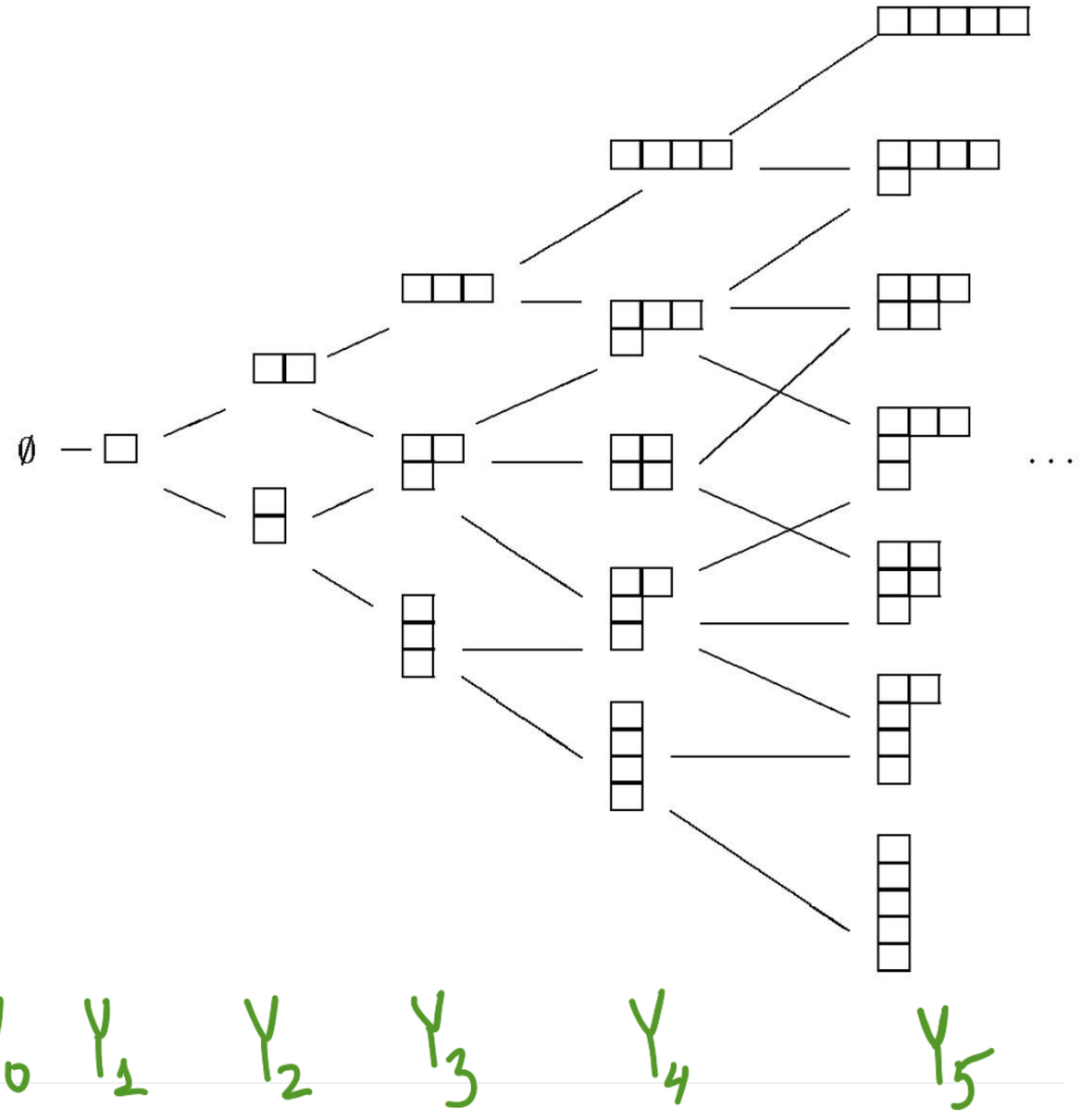
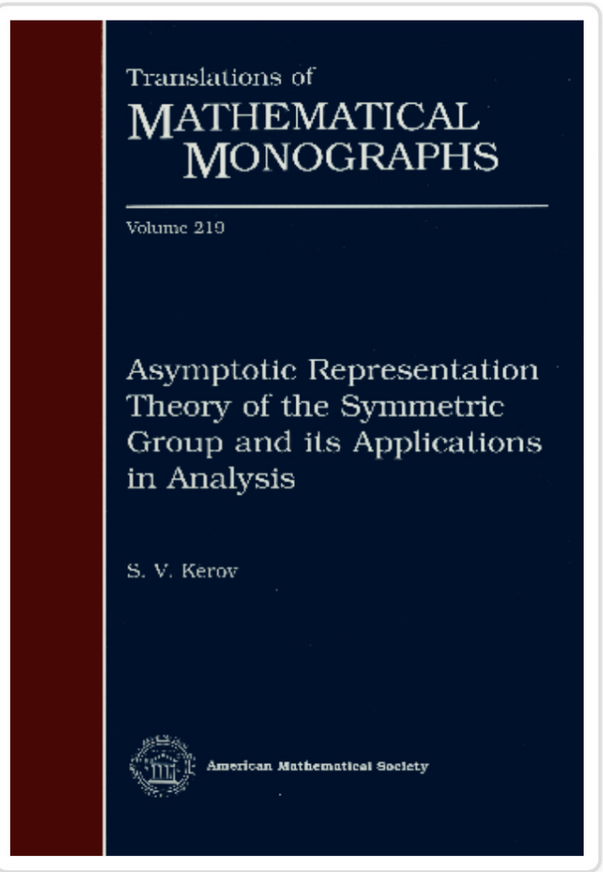
We usually denote partitions / Young diagrams by Greek letters λ, μ, ν and write $\lambda = \mu + \square$ or $\mu \nearrow \lambda$ when they differ by a box

The first levels of the Young graph



Picture from:



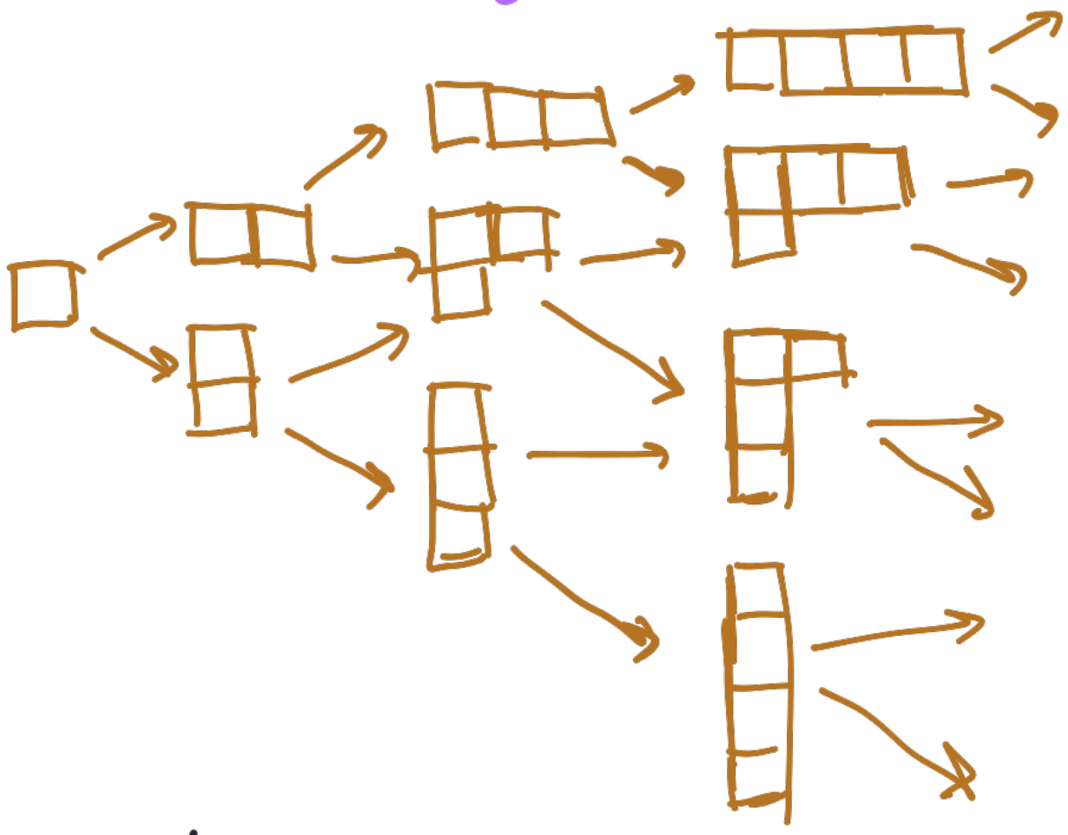
This is a translation of the doctoral thesis (Russian analogue of European „habilitation") of Sergei Kerov. Young graph plays a central role in the book.

The study of the Young graph is a very rich subject in algebraic combinatorics. Some features:

I. Consider the subgraph of hook diagrams of the form $(a+1, 1^b)$

one long row

one long column



Lemma: This subgraph is isomorphic to the Pascal's graph

Proof: Bijection is explicitly given by

$(a+1, 1^b) \longrightarrow a \in \{0, 1, \ldots a+b\}$

in Y_{a+b+1} — in $(a+b)$-th level of the Pascal's graph

Corollary: In the Young's graph

$$\dim(a+1, 1^b) = \#\{\text{paths } \emptyset \to (a+1, 1^b)\} = \binom{a+b}{a}$$

II. The subgraph of two-row diagrams is isomorphic to trajectories of positive random walk with ± 1 steps.

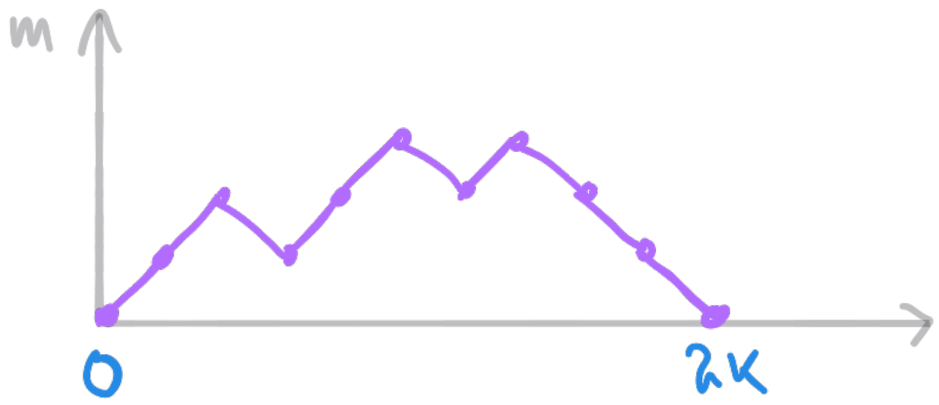
In particular, we have

Lemma: $\dim(K, K) = K\text{-th Catalan number} = \frac{1}{K+1}\binom{2K}{K}$

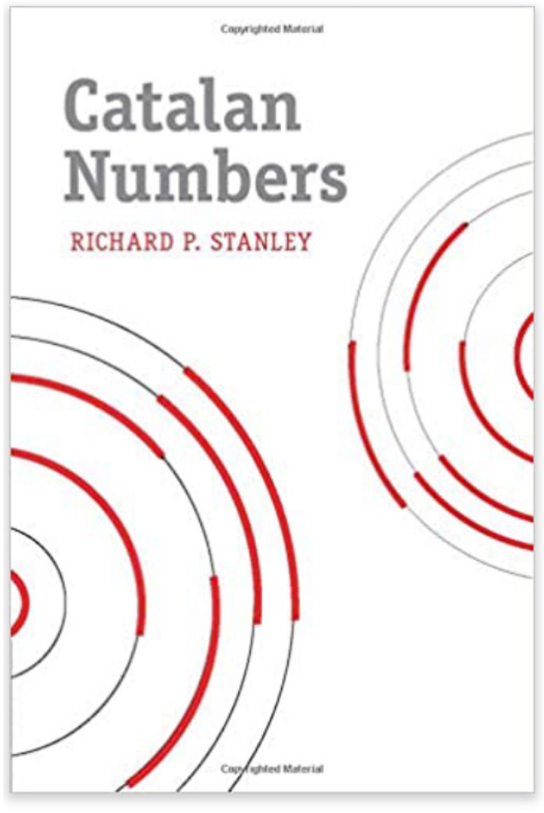
$\#\{\text{paths } \emptyset \to \{(K,K)\} \text{ in } Y\}$

Proof. We identify two-row diagrams (a,b) with non-negative numbers $m := a-b$. Then addition of one box to (a,b) leads to one of the two transitions: either $m \to m+1$ or $m \to m-1$, subject to the constraint that m must remain non-negative.

Hence, paths $\emptyset \to (k,k)$ are in bijection with non-negative trajectories with $(1,1)$ or $(1,-1)$ steps linking $(0,0) \to (2k,0)$ and whose abscissas stay ≥ 0.

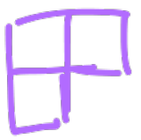
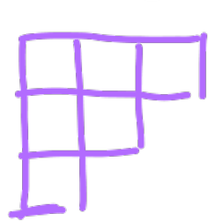


The number of such trajectories is one of the definitions of the Catalan numbers



See Stanley's book for 213 other definitions

The explicit formula can be obtained either by using the reflection principle for random walks or by writing a recurrence and then solving it using generating functions

III. **Theorem 1**: Let Δ_K denote the staircase diagram $(K, K-1, K-2, \ldots 1)$ [K=2  K=3 ]

then paths $\emptyset \to \Delta_K$ in Y are in bijection with **sorting networks** of rank $K+1$.

note shift by 1 ↕

Definition: A sorting network of rank n is a way to represent reverse permutation $(n, \ldots, 3, 2, 1)$ as a product of $\frac{n(n-1)}{2}$ swaps $(i, i+1)$, $0 < i < n$.

minimal possible number

Graphically they are represented as **wiring diagrams**.

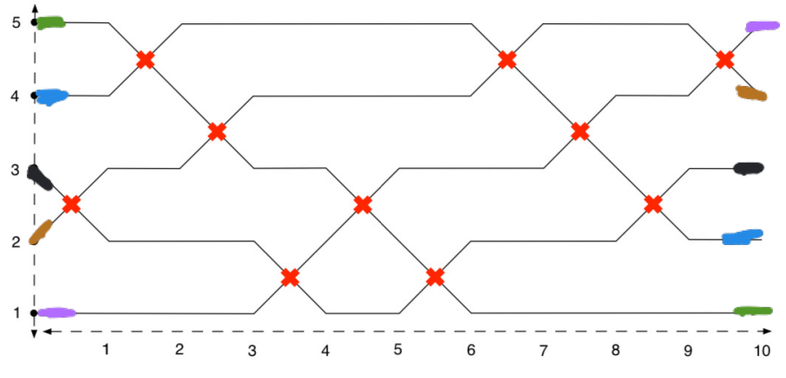


FIGURE 1. Wiring diagram of a sorting network of $\mathfrak{S}_5$ with swap sequence (2,4,3,1,2,1,4,3,2,4). Intersection of two paths at location $(i-1/2, j+1/2)$ indicates a swap at time i between labels at positions j and $j+1$.

Exercise 1: Check theorem 1 for $K=3$ by finding all sorting networks of rank 4 and all paths $\emptyset \to \Delta_3$ in Y.

Theorem 1 is quite complicated and we will not provide a proof here.

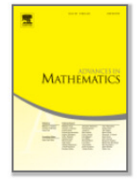
Europ. J. Combinatorics (1984) **5**, 359–372

On the Number of Reduced Decompositions of Elements of Coxeter Groups

RICHARD P. STANLEY*

← First observation of this fact and a proof by showing that the number of elements in both sets is the same

Advances in Mathematics
Volume 63, Issue 1, January 1987, Pages 42-99



Construction of an explicit bijection →

Balanced tableaux

Paul Edelman *, Curtis Greene †

Topics in Algebraic Combinatorics LECTURE NOTES April 4, 2001

The Saga of Reduced Factorizations of Elements of the Symmetric Group

by
A. M. Garsia

← (A bit outdated, but very good otherwise) review of numerous subsequent developments.

Google "Random sorting networks" for fascinating probabilistic part of the story.

IV. Consider the algebra $\mathbb{R}[x,y]$ of polynomials in two variables x and y. Monomial ideal in this algebra are in bijection with $\lambda \in Y$

[Ideal = a linear subspace closed under multiplications by polynomials. Monomid ideal = ideal spanned by some set of monomials]

Correspondence:

1	x	x^2	x^3	x^4	x^5
y	xy	x^2y	x^3y	x^4y	
y^2	xy^2	x^2y^2	x^3y^2	x^4y^2	
y^3	xy^3	x^2y^3	x^3y^3	x^4y^3	
y^4					

$\lambda \in Y$

$\updownarrow$

$\langle$all monomials outside it$\rangle$

Example: $\square \in Y_1 \leftrightarrow \langle x, y \rangle$

Ideals are ordered by inclusion. Under correspondence

$\lambda = \mu + \square \longleftrightarrow$ Ideal $(\lambda) \subset$ Ideal (μ), they are distinct and there are no ideals in between

V: The Young graph is the Hasse diagram of the distributive Young lattice / differential Young POSET.

You do not have to understand any of these words. We just want to illustrate that there are rich combinatorial structures surrounding the Young graph, including many more not mentioned!

Our interest is more probabilistic.

Question: What is the boundary of the Young graph? I.e., what are all coherent systems $(M_0, M_1, M_2, \dots)$ such that M_n is a probability measure on Y_n and for each $\mu \in Y_n$ we have $M_n(\mu) = \sum_{\lambda = \mu + \square} \frac{\dim \mu}{\dim \lambda} M_{n+1}(\lambda)$?

Theorem 2: The extreme coherent systems on Y are parameterized by pairs of real sequences:

$\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$, such that $\sum_{i=1}^{\infty} (\alpha_i + \beta_i) \leq 1$. With the notation $\gamma = 1 - \sum_{i=1}^{\infty} (\alpha_i + \beta_i)$ the corresponding coherent system is:

$$M_n^{(\alpha,\beta)}(\lambda) = \dim \lambda \cdot \det \left[C_{\lambda_i - i + j} \right]_{i,j=1}^{m} \qquad (*)$$

[m can be any number $\geq$ number of rows in λ]

Where $(**)$ $\sum_{k \in \mathbb{Z}} C_k z^k = e^{\gamma z} \cdot \prod_{i \geq 1} \frac{1 + \beta_i z}{1 - \alpha_i z}$, in particular $C_k = 0$ for $k < 0$.

↖ similarity with Week 2 is NOT a coincidence

Exercise 2: Take $\alpha_1 = p$, $\beta_1 = 1-p$, all other parameters $= 0$. Check that the resulting coherent system is supported on hook diagrams (slide 4) and matches the coherent system of i.i.d. Bernoulli's (Weeks 3-4)

There are at least five (perhaps, more) distinct proofs of Theorem 2 in the literature. Why?

Because it is equivalent to several other important statements.

We prove Theorem 2 next week. Today we highlight its important reformulations.

The first (historically) appearance of such statements is in the theory of total positivity pioneered by the research group of Schoenberg

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Total positivity is a large topic, existing since 1930's and more popular recently.

Isaac Jacob Schoenberg

American mathematician

Isaac Jacob Schoenberg was a Romanian-American mathematician, known for his discovery of splines. Wikipedia

Born: April 21, 1903, Galați, Romania

Died: February 21, 1990, Madison, WI

Education: Alexandru Ioan Cuza University

Books: Cardinal Spline Interpolation, Mathematical Time Exposures, Selected Papers: Vol. 1, Selected Papers: Vol. 2

Academic advisor: Issai Schur

In Madison since 1966

Schoenberg studied the following problem:

Take a real sequence $(a_n)_{n\in\mathbb{Z}}$ and consider the convolution transformation (on real sequences):

$$(x_n)_{n\in\mathbb{Z}} \longrightarrow \left(y_n = \sum_k x_k a_{n-k}\right)_{n\in\mathbb{Z}}$$

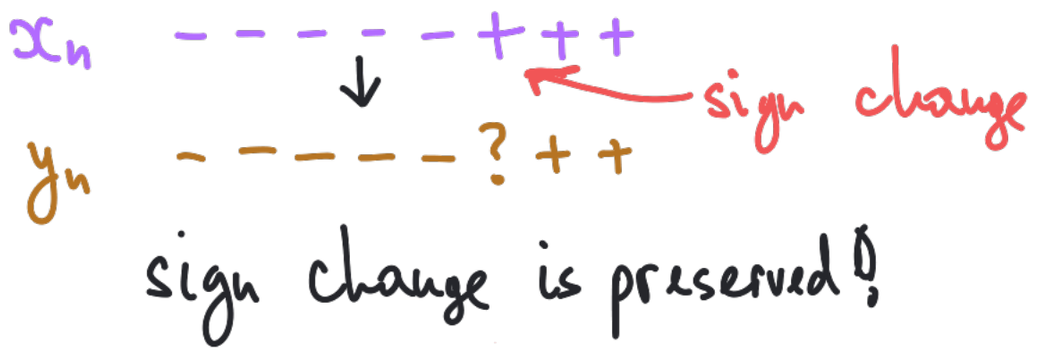
terminology of Polya

Definition: The sequence (a_n) is called variation diminishing if $\sum_n |a_n| < \infty$ and for every bounded (x_n), the number of sign changes in (y_n) is smaller or equal than the number of sign changes in (x_n).

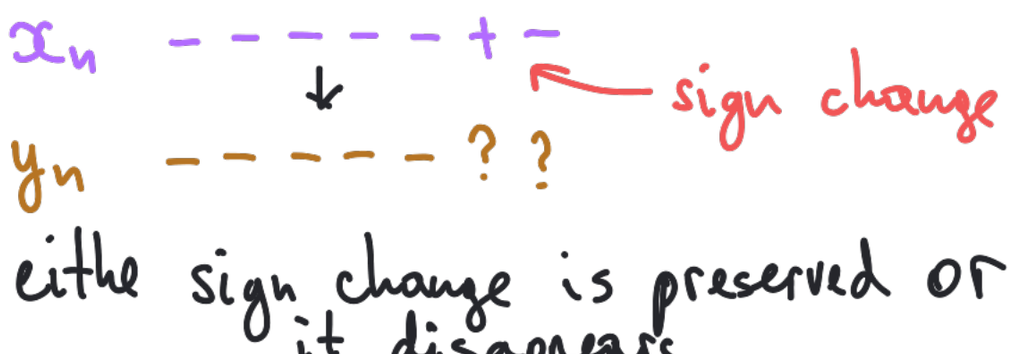
Theorem 3 (Schoenberg; we do not provide a proof). (a_n) is variation diminishing if and only if: $\sum |a_n| < \infty$, and either the matrix $[a_{j-i}]$ or $[-a_{j-i}]$ is totally positive, which means that all its minors (determinants of submatrices) are ≥ 0.

Theorem 4: Assume $a_n = 0$ for $n < 0$, i.e. the sequence is one-sided. For normalization also assume $a_0 = 1$. Then the Toeplitz matrix $[a_{j-i}]$ is totally positive if and only if $a_k = c_k \cdot r^k$ with c_k coming from Slide 11, (**).
[$r > 0$]

Example: $a_0 = 1$, $a_1 = 1$, all other parameters $= 0$. [$\beta_1 = 1$ in Theorem 2]
Then $y_n = x_n + x_{n-1}$. Let us look at the left-most sign change in (x_n) and (y_n) (assume $x_{-\infty} < 0$)



sign change is preserved!



either sign change is preserved or it disappears

Continuing with next sign changes one shows variation diminishing.

Theorem 4 was conjectured by Schoenberg in 1948 and proved in 1951 with final contribution by Edrei

On the Generating Functions of Totally Positive Sequences

Michael Aissen, Albert Edrei, I. J. Schoenberg, and Anne Whitney

PNAS May 1, 1951 37 (5) 303-307; https://doi.org/10.1073/pnas.37.5.303

On the generating functions of totally positive sequences I

Michael Aissen, I. J. Schoenberg & A. M. Whitney

Journal d'Analyse Mathématique **2**, 93–103(1952) | C

On the generating functions of totally positive sequences II

Albert Edrei

Journal d'Analyse Mathématique **2**, 104–109(1952) | Cite this article

Edrei, A. (1953). Proof of a Conjecture of Schoenberg on the Generating Function of a Totally Positive Sequence. *Canadian Journal of Mathematics, 5*, 86-94. doi:10.4153/CJM-1953-010-3

[In fact they found two-sided sequences as well]

Sketch of reduction Theorem 2 => Theorem 4.

- Step 1: check directly that (**) with $\beta_1 = 1$, others $= 0$ and $\delta_1 = 1$, others $= 0$ lead to totally positive sequences.
- Step 2: check that if matrices U and V are totally positive, then so is UV. Use this to show that all (a_k) of Theorem 4 indeed give rise to totally positive sequences.

[Steps 1-2 done by Schoenberg in 1948]

- If other totally positive sequences existed, then (*) would give more (positive!) coherent measures. Additional argument gives extremality. Contradiction. ∎

The second reincarnation of Theorem 2 was 15 years later. E. Thoma essentially reproved it, being not aware about the work of Schoenberg.

Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe

Elmar Thoma

Mathematische Zeitschrift **85**, 40–61(1964) | Cite this article

He was interested in representation theory of $S(\infty) = \bigcup_{n\geq 1} S(n)$

Definition: Take a group G. Its normalized character is a function $\chi: G \to \mathbb{C}$, such that

0. χ is continuous. [when G is discrete, this can be omitted]
1. χ is central: $\chi(ab) = \chi(ba)$, equivalently $\chi(aba^{-1}) = \chi(b)$ for any $a, b \in G$.
2. χ is positive definite: for each k and each $g_1, \dots g_k \in G$. the matrix $[\chi(g_i g_j^{-1})]_{i,j=1}^{k}$ is Hermitian and non-negative semidefinite.
[Means $\chi(g^{-1}) = \overline{\chi(g)}$ and $\forall z_1, \dots z_k \in \mathbb{C}$ $\sum_{i,j=1}^{k} \chi(g_i g_j^{-1}) z_i \bar{z}_j \geq 0$]
3. χ is normalized by $\chi(e) = 1$. [e - identity in the group]

Theorem 5: Extreme normalized characters of the infinite symmetric group $S(\infty)$ are parameterized by $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$, $\beta_1 \geq \beta_2 \geq \dots \geq 0$, $\gamma \geq 0$ such that $\gamma + \sum_i (\alpha_i + \beta_i) = 1$. The value of character $\chi^{(\alpha,\beta)}$ on a permutation $g \in S(\infty)$ is computed as follows:

Write g as a union of disjoint cycles $i_1 \to i_2 \to \dots \to i_k \to i_1$ and let $m_k(g) = \#\{\text{cycles of length } k \text{ in } g\}$.

Then $$\chi^{(\alpha,\beta)}(g) = \prod_{k \geq 2} p_k^{m_k(g)},$$ [note that $m_1(g) = \infty$]

where $$p_k = \sum_{i=1}^{\infty} (\alpha_i)^k + (-1)^{k-1} \sum_{i=1}^{\infty} (\beta_i)^k.$$

Example: $\gamma = 1$, all other parameters $= 0$. Then $p_k = 0$, $k \geq 2$

Hence, $\chi^{(0,0)}(g) = \begin{cases} 1, & g = e, \\ 0, & \text{otherwise.} \end{cases}$ [Check that this is, indeed, an extreme normalized character!]

Proposition: As convex sets {normalized characters of $S(\infty)$} and {coherent systems on Young graph} are isomorphic.

The proof uses some representation-theoretic facts, which we need to review first.

Let G be a finite (or compact) group. [So NOT $S(\infty)$.]

Here is a way to construct a character:

Take a unitary representation of G in a finite-dimensional space V, i.e., a homomorphism $\pi : G \to \mathcal{U}(V)$

[invertible linear transformations preserving a scalar product
= $\dim V \times \dim V$ unitary matrices]

Define $\chi^{\pi}(g) := \dfrac{\mathrm{Trace}\,(\pi(g))}{\dim V} = \dfrac{\mathrm{Trace}\,(\pi(g))}{\mathrm{Trace}\,\pi(e)}$

Let us check that this is a normalized character:

1. $$\chi^{\pi}(gh) = \frac{\mathrm{Trace}\,(\pi(gh))}{\dim V} = \frac{\mathrm{Trace}\,(\pi(g)\pi(h))}{\dim V} = \frac{\mathrm{Trace}\,(\pi(h)\pi(g))}{\dim V} =$$

$$= \frac{\mathrm{Trace}\,(\pi(hg))}{\dim V} = \chi^{\pi}(hg).$$

[We used $\pi(gh)=\pi(g)\,\pi(h)$ and $\mathrm{Trace}\,(AB)=\mathrm{Trace}\,(BA)$]

2. $$\sum_{i,j=1}^{K} \chi^{\pi}(g_i g_j^{-1})\, z_i \bar{z}_j = \frac{1}{\dim V}\,\mathrm{Trace}\left(\sum_{i,j} \pi(g_i)\pi(g_j^{-1})\, z_i \bar{z}_j\right) =$$

$$= \frac{1}{\dim V}\,\mathrm{Trace}\left(\left(\sum_i \pi(g_i) z_i\right)\cdot\left(\sum_i \pi(g_i) z_i\right)^{*}\right) \geq 0.$$

[We used $\pi(g^{-1}) = \pi(g)^{-1} = \pi(g)^{*}$ – unitarity – and the fact that all eigenvalues of operator AA^{*} are ≥ 0.]

3. $$\chi^{\pi}(e) = \frac{\mathrm{Trace}\;\pi(e)}{\dim V} = 1.$$

$\pi(e)$: identical operator

Fact 1: For a finite (or compact) group G the set of all normalized characters is a simplex with extreme points χ^π, where π are irreducible representations [i.e. such that there is no non-trivial $W \subset V$ invariant under all operators $\pi(g)$]

For big groups such as $S(\infty)$ this is more complicated, since most representations are ∞-dimensional and trace is infinite.

Fact 2: For $S(n)$ irreducible representations are parameterized by partitions of n. Hence, combining with Fact 1, we get a bijection

$$\{\text{normalized characters of } S(n)\}$$
$$\updownarrow$$
$$\{\text{probability measures on } Y_n\}$$

Fact 3: Take an irreducible representation of $S(n+1)$ π^λ in space V^λ and restrict it onto $S(n) \subset S(n+1)$.

Then one has a multiplicity-free decomposition in irreducibles:

$$V^\lambda = \bigoplus_{\mu:\ \lambda=\mu+\square} V^\mu \qquad (***)$$

(V^λ: representation of $S(n+1)$; V^μ: representations of $S(n)$; "branching rule")

Corollary:

$$\chi^{\pi^\lambda}\Big|_{S(n)} = \sum_{\mu:\ \lambda=\mu+\square} \frac{\dim(\mu)}{\dim(\lambda)}\, \chi^{\pi^\mu}.$$

This is $\dim(V^\lambda)$, but also $\#\{\text{paths } \emptyset \to \lambda \text{ in } Y\}$ by iterations of $(***)$

Proof of Proposition from Slide 18: The correspondence is constructed by

$$\chi \longleftrightarrow (M_0, M_1, M_2, \dots)_Y$$

(χ: character of $S(\infty)$; $(M_0, M_1, M_2, \dots)_Y$: coherent system on Y)

$$\chi|_{S(n)} = \sum_{\lambda \in Y_n} \chi^{\pi^\lambda} \cdot M_n(\lambda)$$

Facts 1-3 guarantee coherency and bijectivity. ☑

We will not discuss total positivity or representations of $S(\infty)$ any deeper in this class. If you are interested:

Bull. Amer. Math. Soc.
Volume 59, Number 3 (1953), 199-230.

← Previous

On smoothing operations and their generating functions

I. J. Schoenberg