Add-on [for Slide 5]: Let ν_1 and ν_2 be two distinct $S(\infty)$-invariant measures on Ω.

Claim: Then $\exists$ $S(\infty)$-invariant measurable $A \subset \Omega$: $\nu_1(A) \neq \nu_2(A)$

Sketch of the proof: set $\eta = \dfrac{\nu_1 + \nu_2}{2}$.

This is a probability measure, such that both ν_1 and ν_2 are absolutely continuous with respect to it. Hence, by Radon-Nickodym theorem, there are densities $\nu_1(x)$ and $\nu_2(x)$, such that $\nu_1[A] = \int_A \nu_1(x)\, d\eta$; $\nu_2[A] = \int_A \nu_2(x)\, d\eta$.

Moreover, $\nu_1(x)$, $\nu_2(x)$ are a.s. unique.
almost surely with respect to η

Note that for any $\sigma \in S(\infty)$:

$\nu_1[\sigma A] = \int_{\sigma A} \nu_1(x)\, d\eta \underset{\text{by } \sigma\text{-invariance of } \eta}{=} \int_A \nu_1(\sigma x)\, d\eta$. Hence, $\nu_1(\sigma x)$ is also a density for ν_1. Therefore, by uniqueness $\nu_1(x) = \nu_1(\sigma x)$ a.s.

Conclusion: $\vartheta_1(x)$ and $\vartheta_2(x)$ are a.s. $S(\infty)$-invariant.

If $\vartheta_1 \neq \vartheta_2$, then either $\{x \mid \vartheta_1(x) \leq \vartheta_2(x) - \varepsilon\}$ or $\{\vartheta_1(x) \geq \vartheta_2(x) + \varepsilon\}$ has positive η-measure for some $\varepsilon > 0$.

Then this gives the $S(\infty)$-invariant set, over which the integrals of $\vartheta_1(x)$ and $\vartheta_2(x)$ with respect to η differ at least by $\varepsilon \cdot \eta[\text{set}]$. ∎