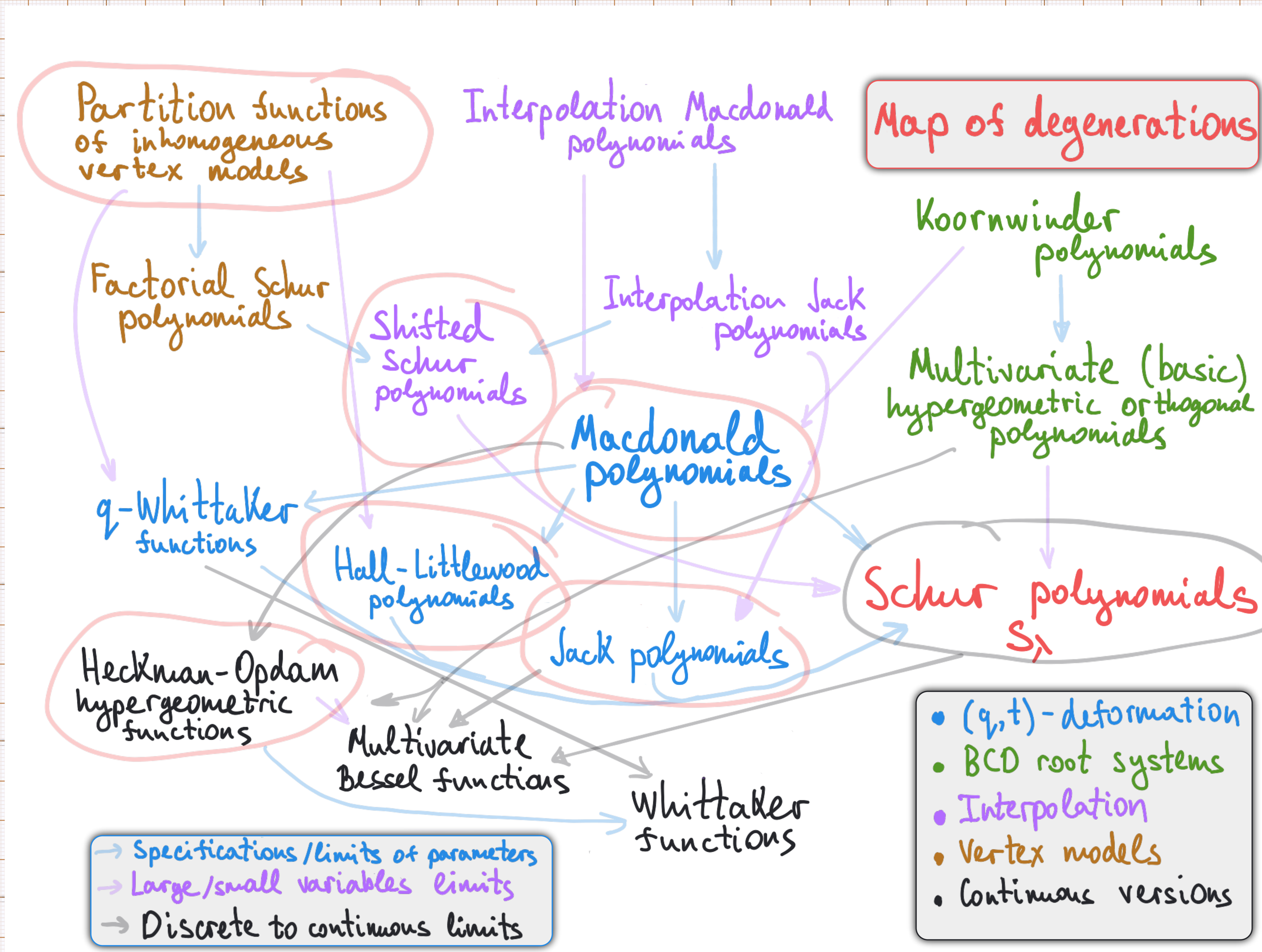
Math 740 Final Review



General features

A) $\lambda \in \mathbb{Y}$ $\quad S_\lambda\,,\ P_\lambda(\cdot\,; q, t),\ J_\lambda(\cdot\,;\theta)$

$m_\lambda\,,\ h_\lambda\,,\ e_\lambda$

B) Stability

$$P_\lambda(x_1, \dots x_N)$$

$$\underline{P_\lambda(x_1, \dots, x_{N-1}, 0)} = \underline{P_\lambda(x_1, \dots x_{N-1})}$$

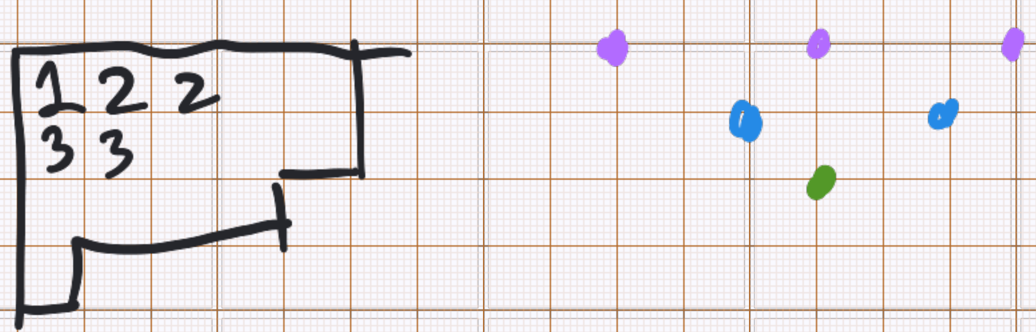
C) They form linear basis in $\Lambda_N, \Lambda, \Lambda^*_N, \Lambda^*$

Central question: How to expand one family of sym. polynomials in the basis given by another family?

D) For each case ($s_\lambda, P_\lambda, s^*_\lambda, J_\lambda, \dots$) the transitio from this family and monomial functions m_λ is known

Combinatorial formula

$$(\text{Polynomial})_\lambda (x_1, \dots x_N) = \sum_{SSYT} (\dots) =$$



$$= \sum_{GT\text{-patterns}} (\dots)$$

E) D is based on branching rules

E.g. $$s_\lambda(x_1, \dots x_N) = \sum_{\mu \prec \lambda} s_\mu(x_1, \dots x_{N-1}) \cdot x_N^{|\lambda|-|\mu|}$$

F) All of them had a version of Cauchy identity

E.g. for Macdonald polynomials

$$\sum_\lambda P_\lambda(x_1, \dots) Q_\lambda(y_1, \dots) = \prod_{i,j=1}^{\infty} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}$$

G) All of them had Pieri rules → rules for multiplication by 1-row or 1-column polynomials from the same family

E.g. for shifted Schur functions

$$s^*_\lambda \cdot \Big(\sum_i x_i - |\lambda| \Big) = \sum_{\mu = \lambda + \square} s^*_\mu$$

"1-box polynomial"

Littlewood-Richardson coefficient

More general formulas $\quad s_\lambda s_\mu = \sum_\nu c^\nu_{\lambda\mu} s_\nu$

H) They are eigenfunctions of difference / differential operators

e.g. for Macdonald polynomials $\mathcal{D} = \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q,i}$

$$\mathcal{D}\, P_\lambda(x_1, \ldots x_n; q, t) = \left(\sum q^{\lambda_i} t^{N-i}\right) P_\lambda(x_1, \ldots x_N; q, t)$$

Operators are simpler than polynomials

I) There are special points, where the value is explicit and given by a multiplicative formula

E.g. Weyl dimension formula

$$S_\lambda(1^N) = \prod_{i<j} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}$$

Applications:

1) Characters of representations of various groups

$$S_n, \quad U(N) \; [GL(N;\mathbb{R}/\mathbb{C})], \quad GL(N;q)$$

2) Fruitful combinatorics: counting plane partitions or tilings or non-intersecting paths; bijections; Robinson-Schensted-Knuth correspondence

3) Integrable Probability: Help to compute expectations of random variables describing complicated stochastic systems. Giving access to asymptotics.

4) Analysis: computations of sums and integrals (e.g. Selberg integral)