Math 740 Shifted Schur functions 2 Lecture 24

Today we discuss more advanced properties of s^*_μ

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1) s^*_μ in representation theory of symmetric groups.

Recall that irreducible representations of S_n are parameterized by partitions of n, $\lambda \in Y_n$ [see Lecture 7]

Take $m < n$ and consider the embedding $S_m \subset S_n$ as the subgroup which fixes $m+1, m+2, \ldots, n$.

Decomposition $$\pi^\lambda \big|_{S_m} = \bigoplus_{\mu \in Y_m} c^\lambda_\mu \, \pi^\mu$$

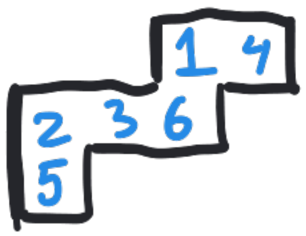
irreducible rep. of S_n; restriction; direct sum; multiplicities; irreducible rep. of S_m

How do you find coefficients c^λ_μ ?

Theorem: $c^\lambda_\mu = \#\{\text{standard Young tableaux of shape } \lambda/\mu\}$

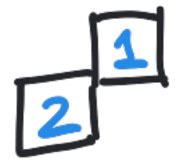
$\parallel$

fillings of $n-m$ boxes of λ/μ by $1, \dots, n-m$ growing along rows and columns



$\lambda = (4,3,1)$

$\mu = (2)$



$\lambda = (2,1)$

$\mu = (1)$

$\parallel$

$\dim \lambda/\mu$

Exercise: Prove the theorem by following the plan:

- Identify representations with characters
- Use Lecture 7, Slide 8 to match character values with transition coefficients between s_λ and p_ν
- Use $p_1 s_\lambda = \sum_{\mu = \lambda + \square} s_\mu$ to get SYT

this identity is symmetric functions version of multiplicity 1 branching of irrep of S_{n+1} under restriction to S_n

OK, but how do you count SYT of skew shape λ/μ?

Theorem 1: $$\frac{\dim \lambda/\mu}{\dim \lambda} = \frac{s^*_\mu(\lambda)}{(|\lambda| \downarrow |\mu|)} \quad (*)$$

explicit, e.g., by hook formula

falling factorial

I) $\frac{\dim \lambda/\mu}{\dim \lambda} \dim \mu$ is relative dimension of $C^\lambda_\mu \pi^\mu$ in π^λ

II) The formula suggests to treat $\frac{\dim \lambda/\mu}{\dim \lambda}$ as a function of λ for some fixed μ

III) The formula was used by Vershik-Kerov and Okounkov-Olshanski for identification of all irreducible characters of the infinite symmetric group S_∞ by sending $\lambda \to \infty$ in $(*)$.

The proof is based on the Pieri rule for s^*_μ

Lemma: $s^*_\mu(p_1 - |\mu|) = \sum_{\lambda=\mu+\square} s^*_\lambda$

Σx_i is symmetric and also shifted-symmetric!

Proof: Restrict to large N number of variables and set $g(x_1,\ldots x_N) = s^*_\mu(x_1,\ldots x_N)\left(\sum_{i=1}^N x_i - |\mu|\right) - \sum_{\lambda=\mu+\square} s^*_\lambda(x_1,\ldots x_N)$

g is shifted-symmetric and of degree $\le|\mu|$

[because the degree $|\mu|+1$ part cancelled by Pieri rule for s_μ]

If we show $g(\nu)=0$ for all $\nu\in\mathbb{Y}: |\nu|\le|\mu|$,
then $g\equiv 0$ by the argument of Lecture 23, Slide 8

- If $|\nu|<|\mu|$, then $s^*_\mu(\nu)=0$, $s^*_\lambda(\nu)=0 \Rightarrow g(\nu)=0.$
- If $|\nu|=|\mu|$, then $\left(\sum_{i=1}^N \nu_i - |\mu|\right)=0$, $s^*(\nu)=0 \Rightarrow g(\nu)=0.$

Proof of Theorem 1: $|\lambda|=n$, $|\mu|=m$, apply Lemma $n-m$ times

We get: $s^*_\mu (p_1-m)(p_1-m-1)\dots(p_1-n+1) = \sum_{\nu : |\nu|=n} \dim \nu/\mu \cdot s^*_\nu$

Now evaluate both sides on λ. For RHS use interpolation property [Lecture 23, Slide 6]

We get: $s^*_\mu(\lambda)\cdot(n-m)! = \dim \lambda/\mu \quad s^*_\lambda(\lambda)$

Choosing $\mu=(0)$, we get $s^*_\lambda(\lambda) = \dfrac{n!}{\dim\lambda}$ ← [was already proven by another method in Lecture 23]

Hence, $s^*_\mu(\lambda)\,\dfrac{(n-m)!}{n!} = \dfrac{\dim\lambda/\mu}{\dim\lambda}$ $\blacksquare$

2) S^*_μ in representation theory of unitary groups

unitary group
$\downarrow$
its Lie algebra
$\downarrow$
its universal enveloping algebra
$\downarrow$
the center Z

[We encountered it previously in Lecture 13]

Harish-Chandra isomorphism $Z \to \Lambda^*(N)$

$A \longrightarrow$ function of labels $\lambda_1 \geq \dots \geq \lambda_N$ of irreducible
$\cap$
Z
representations given by the (diagonal) action of A in corresponding representation.

[As we saw in Lecture 13 and HW4, Problem 1, this is actually a symmetric function of $\{\lambda_i + N - i\}$]

Preimages of $S^*_\mu(x_1, \dots x_n)$ under Harish Chandra isomorphism give a remarkable linear basis of the center Z. These preimages are called

quantum immanants

[immanants are certain generalizations of determinant and permanent of a matrix]

Corollary: Most of the results about S^*_μ imply something about the structure of the center Z

More prosaic appearance of s^*_μ in representation theory of $\mathcal{U}(N)$:

Theorem 2 [Binomial theorem for $\mathcal{U}(N)$ or $GL(N)$]

(**)
$$\frac{S_\lambda(1+x_1,\dots,1+x_N)}{S_\lambda(1^N)} = \sum_\mu \frac{s^*_\mu(\lambda_1,\dots,\lambda_N)\, S_\mu(x_1,\dots x_N)}{\prod_{(i,j)\in\mu}(N+j-i)}$$

I) This is Taylor series expansion of normalized characters of the group $\mathcal{U}(N)$ near identity

II) Applied by Vershik-Kerov and Okounkov-Olshanski for finding all irreducible characters of $\mathcal{U}(\infty)$: These are $N\to\infty$ limit of (**) on $(x_1,\dots x_K, 0, 0, \dots)$.

Proof of Theorem 2: From HW 2, Problem 5 we know

$$(***) \qquad s_\lambda(1+x_1,\ldots,1+x_N) = \sum_{\mu\subset\lambda} \det\left[\binom{\lambda_i+N-i}{\mu_j+N-j}\right]_{i,j=1}^{N} s_\mu(x_1,\ldots,x_N).$$

$$\binom{\lambda_i+N-i}{\mu_j+N-j} = \frac{(\lambda_i+N-i \downarrow \mu_j+N-j)}{(\mu_j+N-j)!}$$

and also

$$s_\lambda(1^N) = \prod_{i<j}\left(\lambda_i+N-i-(\lambda_j+N-j)\right)\cdot\prod_{i=1}^{N-1}\frac{1}{(N-i)!}$$

It remains to note that

$$\prod_{(i,j)\in\mu}(N+j-i) = \prod_{i=1}^{N}\frac{(\mu_i+N-i)!}{(N-i)!}$$

Plugging everything into (***) we get (**). $\blacksquare$

Shifted Schur functions admit generalizations in Macdonald hierarchy. [Knop, Sahi, Okounkov, Olshanski]

- Shifted Jack polynomials $\mathcal{J}^*_\lambda(x_1, \dots x_N; \theta)$
  symmetric in $x_i - \theta i$; interpolation property in λ_i;

Binomial formula: $$\frac{J_\lambda(1+x_1, \dots, 1+x_N; \theta)}{J_\lambda(1^N; \theta)} = \sum_\mu \frac{\mathcal{J}^*_\mu(\lambda; \theta)\, \tilde{J}_\mu(x_1, \dots x_N; \theta)}{\prod_{(i,j)\in\mu} (N + j - \theta i)}$$

- Interpolation Macdonald polynomials $P^*_\lambda(x_1, \dots, x_N; q, t)$
  symmetric in $x_i t^{-i}$; interpolation property in $q^{\lambda_1}, \dots q^{\lambda_N}$;

Binomial formula: $$\frac{P_\lambda(x_1, x_2 t^{-1}, \dots, x_N t^{1-N}; q, t)}{P_\lambda(1, t^{-1}, \dots, t^{1-N}; q, t)} = \sum_\mu \frac{P^*_\mu(q^\lambda; q, t)}{P^*_\mu(q^\mu; q, t)} \cdot \frac{P^*_\mu(x_1, \dots x_N; q, t)}{P_\mu(1, \dots, t^{1-N}; q, t)}$$

All of them are useful in asymptotic representation theory