Math 740 Shifted Schur functions Lecture 23

So far: homogeneous symmetric polynomials

This week: their remarkable inhomogeneous versions

Definition:

Okun'kov, A. Yu.; Ol'shanskij, G. (A. Okounkov, G. Olshanski)
Shifted Schur functions. (English. Russian original) Zbl 0894.05053
St. Petersbg. Math. J. 9, No. 2, 239-300 (1998); translation from Algebra Anal. 9, No. 2, 73-146 (1997).

$$S^*_\mu(x_1, \dots x_N) = \frac{\det\left[(x_i + N - i \downarrow \mu_j + N - j)\right]_{i,j=1}^{N}}{\det\left[(x_i + N - i \downarrow N - j)\right]_{i,j=1}^{N}}$$

$\mu \in \mathbb{Y}$

where falling factorial power is

$$(x \downarrow n) = x(x-1) \cdot \ldots \cdot (x - n + 1)$$

$$(x \downarrow 0) = 1$$

Basic properties:

$$S^*_\mu(x_1,\dots x_N) = \frac{\det\left[(x_i+N-i \downarrow \mu_j+N-j)\right]_{i,j=1}^N}{\det\left[(x_i+N-i \downarrow N-j)\right]_{i,j=1}^N}$$

- Numerator is skew-symmetric in variables $\{x_i+N-i\}$ of degree $|\mu| + \frac{N(N-1)}{2}$
- Denominator is $\prod_{i<j}\left[(x_i+N-i)-(x_j+N-j)\right]$
- S^*_μ is symmetric in $\{x_i+N-i\}$ of degree $|\mu|$
- $S^*_\mu(x_1,\dots x_N) = S_\mu(x_1,\dots x_N) + (\text{terms of lower degree})$

  Inhomogeneous deformation of Schur polynomials
- At $N=1$: $\underset{S^*_{(1)}(x)}{x}$, $\underset{S^*_{(2)}(x)}{x(x-1)}$, $\underset{S^*_{(3)}(x)}{x(x-1)(x-2)}$, $\underset{S^*_{(4)}(x)}{x(x-1)(x-2)(x-3)}$, ...

At $N=1$ we observe interpolation property

Lemma: $s^*_{(k)}(x) = x(x-1)\cdot\ldots\cdot(x-k+1)$ is a unique polynomial of degree k, with leading coefficient 1, and such that its values at $x=0,1,2,\ldots,k-1$ all vanish.

Proof: A monic degree k polynomial $x^k + a_{k-1}x^{k-1} + \ldots + a_0$ is uniquely fixed by its values at (arbitrary) k points ■

These polynomials show up in interpolation problem:

Given $f(0), f(1), \ldots, f(n)$, find a polynomial $g(x)$ of degree $\leq n$, such that $g(x) = f(x)$ for $x \in \{0,1,\ldots n\}$.

More generally we can try to construct a degree n polynomial approximation of a function by its values in $(n+1)$ points $x_0, x_1, \ldots, x_n$.

Lemma: Given $n+1$ values $g(0), \dots g(n)$, there is a unique polynomial $g(x)$ of degree $\le n$, taking these values.

Proof: We have $(n+1)$ non-degenerate linear equations on $(n+1)$ coefficients of polynomial $g(x)$. $\blacksquare$

In practice, there are two popular algorithms.

- Lagrange interpolation

$$g(x) = \sum_{i=0}^{n} g(i) \prod_{j \neq i} \frac{(x-j)}{(i-j)}$$

Advantages: explicit and simple formula

Disadvantages: numerically unstable; when n is increased, need to recompute from scratch.

- Newton interpolation

$$g(x) = \sum_{k=0}^{n} g_k \cdot x(x-1) \dots (x-k+1)$$

Advantages: better numerics; only one new coefficient added when we increase n (others stay the same)

Disadvantages: Less explicit g_k — either through divided differences or recurrently.

Recurrent Newton interpolation: $g^{(n)}(x) = \sum_{k=0}^{n} g_k \cdot x(x-1)\cdot\ldots\cdot(x-k+1)$

Start from $g^{(0)}(x) = g(0)$. Given $g^{(n)}(x)$, define:

$$g^{(n+1)}(x) = g^{(n)}(x) + \left(g(n+1) - g^{n}(n+1)\right) \frac{x(x-1)\ldots(x-n)}{n!}$$

Outcome: $g^{(n)}(x)$ has desired values at $0, 1, \ldots, n$.

Conclusion: $N=1$ versions of S^*_μ are useful in (Newton) polynomial interpolation because of their vanishing property.

Coming next: Same is true for $N>1$ for symmetric polynomial interpolation.

„Interpolation property of shifted Schur polynomials"

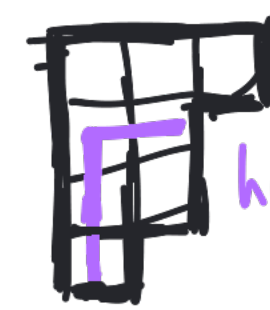
Theorem: Fix N for $\lambda \in \mathbb{Y} \cap GT_N$ denote $s^*_\mu(\lambda) = s^*_\mu(\lambda_1, \lambda_2, \dots \lambda_N)$.
For each $\mu \in \mathbb{Y} \cap GT_N$, $s^*_\mu(x_1, \dots x_N)$ is a unique polynomial, such that:

A: s^*_μ is symmetric in $\{x_i + N - i\}$ „shifted-symmetric"

B: s^*_μ has degree $|\mu|$

C: $s^*_\mu(\lambda) = 0$ for all λ such that $|\lambda| \leq |\mu|$, $\lambda \neq \mu$

D: $s^*_\mu(\mu) = H(\mu) = \prod_{(i,j)\in\mu} h(i,j)$ hook length



$h(2,1) = 4$

Proof: We start by showing that s^*_μ as defined on slide 1 satisfies A-D. We already know A and B.

For C we are going to prove an even stronger statement:

$s^*_\mu(\lambda) = 0$ whenever $\mu_i > \lambda_i$ for at least one $i \in \{1, \dots n\}$

Note that $(a \downarrow b) = 0$ if $b > a \geq 0$

Consider the matrix $\left[(\lambda_i + N - i \downarrow \mu_j + N - j) \right]_{i,j=1}^{N}$

If $\mu_k > \lambda_k$, then also $\mu_j > \lambda_i$, $j = 1, \ldots k$, $i = k, \ldots N$.

Hence, the matrix has $k \times (N-k+1)$ corner of zeros and its determinant vanishes.

For D we need to compute the determinant

$$\det \left[(\mu_i + N - i \downarrow \mu_j + N - j) \right]_{i,j=1}^{N} = \prod_{i=1}^{N} (\mu_i + N - i)!$$

(because the matrix is triangular)

Hence, $$S^*_\mu(\mu) = \frac{\prod_{i=1}^{N} (\mu_i + N - i)!}{\prod_{i<j} (\mu_i + N - i - (\mu_j + N - j))} = H(\mu)$$

HW3, Problem 1.

Next, we need to show that s^*_μ is uniquely determined by A-D. Let f_μ be a polynomial satisfying A-D and set $g = f_\mu - s^*_\mu$.

Then g is a shifted-symmetric polynomial of degree $k = |\mu|$ and such that $g(\lambda) = 0$ for all $\lambda : |\lambda| \le k$.

Note that s^*_ν, $|\nu| \le k$, form a linear basis in the vector space of such polynomials (because their highest degree component is s_ν). Thus, we can expand

$$g = \sum_{\lambda : |\lambda| \le k} c^\lambda \cdot s^*_\lambda(x_1, \ldots x_N)$$

by C, D, inductively in $|\lambda|$

Plug $x_1 = \lambda_1, \ldots, x_N = \lambda_N$ to get $0 = c^\lambda \cdot H(\lambda)$

Hence, $g \equiv 0$ and $f_\mu = s^*_\mu$, as desired. $\blacksquare$

Recall that previously all symmetric polynomials had $N=\infty$ variables versions. How about s_μ^* ?

They belong to a different algebra!

Definition: Λ^* is a projective limit of algebras $\Lambda^*(N)$ of shifted-symmetric polynomials in x_i, $1 \leq i \leq N$ (= symmetric in $\{x_i+N-i\}$ = symmetric in $\{x_i-i\}$) with respect to projections $\Lambda^*(N) \xrightarrow{p_N} \Lambda^*(N-1)$ setting $x_N \to 0$ and in the category of graded (by degree) algebras.

Element of Λ^* is a sequence $(\underset{\Lambda^*(1)}{f_1}, \underset{\Lambda^*(2)}{f_2}, \underset{\Lambda^*(3)}{f_3}, \ldots)$ with $p_K(f_K) = f_{K-1}$ and $\sup_K \deg(f_K) < \infty$

Exercise: $\Lambda^* = \mathbb{R}[p_1^*, p_2^*, \ldots]$ with $p_K^* = \sum_{i \geq 1} \left[(x_i - i)^K - (-i)^K \right]$

Stability theorem: $s^*_\mu \in \Lambda^*(N)$ agree with projections:

$$s^*_\mu(x_1, \dots x_{N-1}, 0) = \begin{cases} s^*_\mu(x_1, \dots x_{N-1}), & \ell(\mu) \le N-1 \\ 0, & \text{otherwise.} \end{cases}$$

Hence, $(0, 0, \dots, s^*_\mu(x_1, \dots x_N), s^*_\mu(x_1, \dots x_{N+1}), \dots)$ defines a shifted-symmetric function $s^*_\mu \in \Lambda^*$.

Proof:

$$s^*_\mu(x_1, \dots x_N) = \frac{\det\left[(x_i + N - i \downarrow \mu_j + N - j)\right]_{i,j=1}^N}{\det\left[(x_i + N - i \downarrow N - j)\right]_{i,j=1}^N} \qquad (*)$$

Assume $\mu_N = 0$ and set $x_N = 0$.

The last row is $(0, 0, \dots, 1)$; from the i-th row we can factor $(x_i + N - i)$, for each $1 \le i \le N-1$.

Remains to pass to $(N-1) \times (N-1)$ minors in $(*)$ to get $(N-1)$ expression.

Next, assume $\mu_N > 0$ and set $x_N = 0$.

Numerator in $(*)$: last row is $(0, 0, \dots, 0) \Rightarrow \det = 0$

Denominator in $(*)$: $\prod_{i<j} (x_i - x_j - i + j)$ does not vanish identically when $x_N = 0$

Therefore, the ratio $(*)$ vanishes, as desired $\blacksquare$

Most properties of s_λ extend to s^*_λ with slight modifications

- Combinatorial formula (inhomogeneous)
- Jacobi-Trudy formula (with shifts)
- Involution of Λ^* mapping s^*_λ to $s^*_{\lambda'}$
- Explicit generating functions for $h^*_k = s^*_{(k)}$ and $e^*_k = s^*_{(1^k)}$
  (Need to compute $\sum \frac{h^*_k}{(u\downarrow k)}$ and $\sum \frac{e^*_k}{(u\downarrow k)}$)

See Okounkov-Olshanski article for the details.