Math 740 Hall-Littlewood Polynomials Lecture 22

$P_\lambda(x_1, \dots x_N; 0, t)$. $q=0$ version of Macdonald polynomials

- $t=0$: Schur polynomials [Characters of S_n and $U(n)$]
- $t=1$: Monomial symmetric functions
- $t=-1$: Schur Q-functions [Projective characters of S_n]

[$\rho: S_n \to GL(V)$: $\rho(uv) = \rho(u) \cdot \rho(v) \cdot \mathrm{const}(u,v)$]

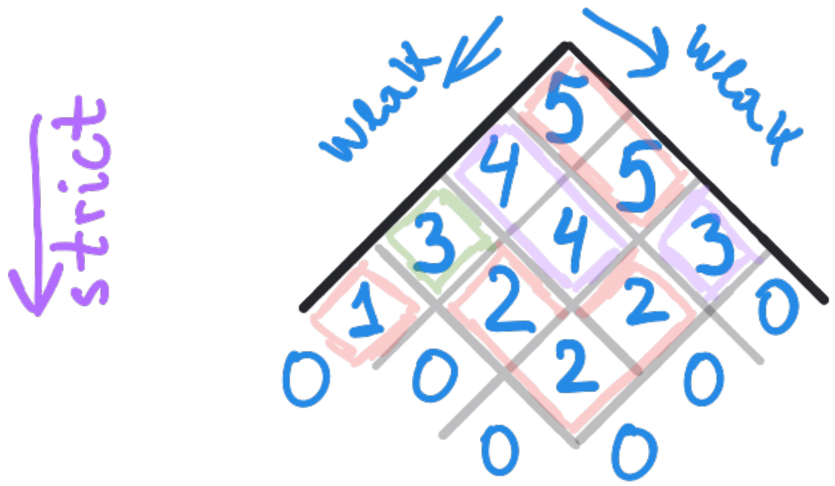
- $t = p^{-1}$: A) Representation theory and random matrices over p-adic numbers

  B) Representation theory and random matrices over finite field with p elements

  C) p-groups [order of each element is p^k]

- Lead to exciting enumerative combinatorics

Here is an example. **Strict plane partition**



- Finitely many >0 integers
- Weakly decrease in ↙ ↘ directions
- Strictly decrease in ↓ direction

$\mathrm{Vol}(\pi)$ = Sum of numbers

[$5+5+4+4+3+3+2+2+2+1 = 31$]

by sharing edges

$K(\pi)$ = number of connected components of equal numbers

[1 component of 5; 4; 2; 1
2 components of 3) $\Rightarrow K=6$]

Theorem: (Strict version of Macmahon's formula)

$$\sum_{\text{strict } \pi} 2^{K(\pi)} q^{\text{Volume}(\pi)} = \prod_{n\geq 1} \left(\frac{1+q^n}{1-q^n}\right)^n$$

This is quite recent:

Journal of High Energy Physics

BKP plane partitions

Omar Foda[1] and Michael Wheeler[1]

Published 19 January 2007 • Published under licence by IOP Publishing Ltd

Journal of High Energy Physics, Volume 2007, JHEP01(2007)

The Shifted Schur Process and Asymptotics of Large Random Strict Plane Partitions

Mirjana Vuletić ✉ Author Notes

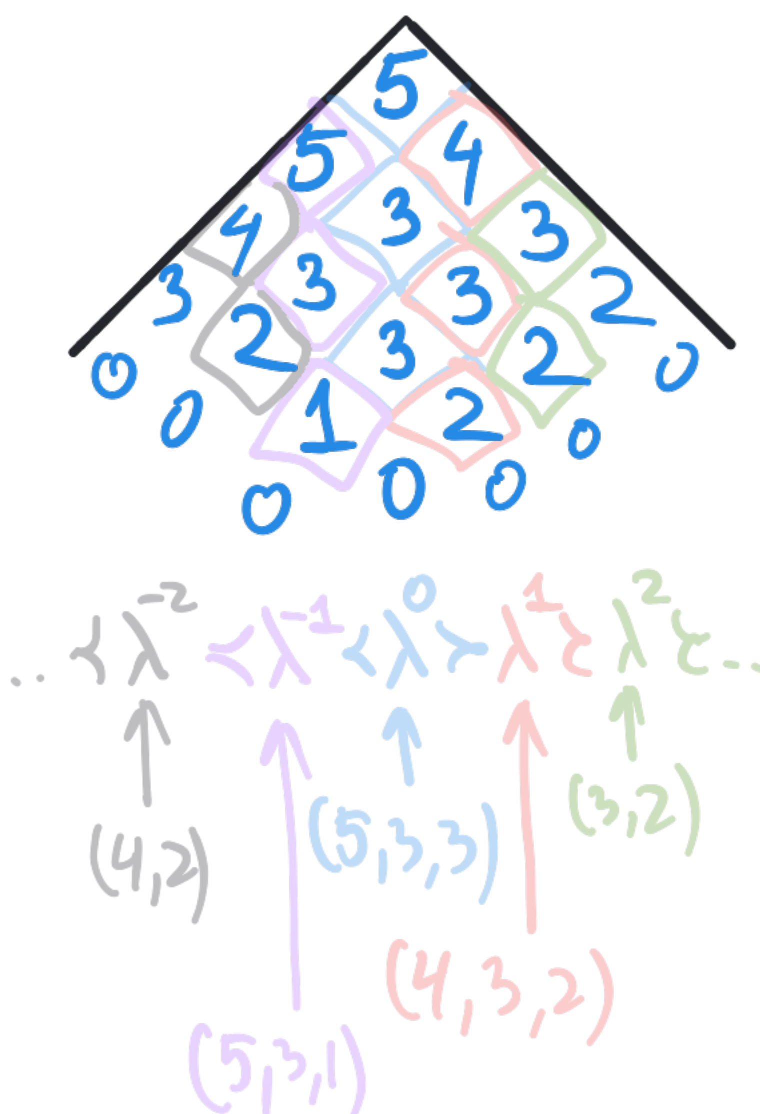
International Mathematics Research Notices, Volume 2007, 2007, rnm043,

https://doi.org/10.1093/imrn/rnm043

Published: 01 January 2007 **Article history** ▾

Sketch of the proof. We start by providing another proof of Macmahon theorem (from the last slide of Lecture 4) which claims

$$\sum_{\substack{\text{all plane partitions} \\ \text{[do not have to be strict]}}} q^{\mathrm{Vol}(\pi)} = \prod_{n \geq 1} (1-q^n)^{-n}$$



Plane partition π

$\parallel$

infinite sequence of interlacing partitions

$$\cdots \prec \lambda^{-n} \prec \cdots \prec \lambda^{-1} \prec \lambda^0 \succ \lambda^1 \succ \cdots \succ \lambda^n \succ \cdots$$

$$\mathrm{Vol}(\pi) = \sum_{n=-\infty}^{\infty} |\lambda^n| =$$

$$= \sum_{n=1}^{\infty} \left(n - \tfrac{1}{2}\right)\left(|\lambda^{n-1}| - |\lambda^n|\right) +$$

$$+ \sum_{n=1}^{\infty} \left(n - \tfrac{1}{2}\right)\left(|\lambda^{-n-1}| - |\lambda^{-n}|\right)$$

Lemma: $$\sum_{\pi} q^{\mathrm{vol}(\pi)} = \sum_{\lambda} S_\lambda^2\left(q^{\frac{1}{2}}, q^{3/2}, q^{5/2}, q^{7/2}, \dots\right)$$

Proof: We expand each S_λ by the combinatorial formula. For the first one we get sum over sequences

$\lambda = \lambda^0 \succ \lambda^1 \succ \lambda^2 \succ \dots$ of weights

$$q^{\frac{1}{2}(|\lambda^0| - |\lambda^1|) + \frac{3}{2}(|\lambda^1| - |\lambda^2|) + \dots}$$

For the second one we get sum over sequences

$\lambda = \lambda^0 \succ \lambda^{-1} \succ \lambda^{-2} \succ \dots$ of weights

$$q^{\frac{1}{2}(|\lambda^0| - |\lambda^{-1}|) + \frac{3}{2}(|\lambda^{-1}| - |\lambda^{-2}|) + \dots}$$

Remains to identify pairs of sequences with plane partitions. $\square$

Now we use Cauchy identity:

$$\sum_{\text{plane partitions}} q^{\mathrm{Vol}(\pi)} = \prod_{i,j\geq 1} \left(1-q^{(i-\frac{1}{2})+(j-\frac{1}{2})}\right)^{-1} =$$

$$= \prod_{i,j\geq 1} \left(1-q^{i+j-1}\right)^{-1} \underset{n=i+j-1}{\overset{\uparrow}{=}} \prod_{n\geq 1} \left(1-q^n\right)^{-n}$$

MacMahon's formula is proven again!

Next step is to repeat the argument for Hall-Littlewood polynomials:

this is not Macdonald's q! [which is set to 0]

$$\sum_{\lambda} P_\lambda(q^{\frac{1}{2}}, q^{3/2}, \ldots; 0, t)\, Q_\lambda(q^{1/2}, q^{3/2} \ldots; 0, t)$$

$$= \prod_{i,j\geq 1} \frac{1-t\,q^{i+j-1}}{1-q^{i+j-1}}$$

[Cauchy identity for Hall-Littlewoods]

One should futher expand P_λ and Q_λ by combinatorial formula getting a weighted sum over plane partitions

$$\sum_{\text{plane partitions}} \text{weight}(t;\pi)\cdot q^{\mathrm{Vol}(\pi)} = \prod_{n\geq 1}\left(\frac{1-tq^n}{1-q^n}\right)^n \qquad (*)$$

$n=i+j-1$ in double product

Exercise: Using formulas from HW 5 find the expression for $\text{weight}(t;\pi)$. Further, show that

$$\text{weight}(-1;\pi) = \begin{cases} 2^{k(\pi)}, & \text{if } \pi \text{ is strict;} \\ 0, & \text{if } \pi \text{ is not strict.} \end{cases}$$

[This should be similar to HW5, Problem 4]

Setting $t=-1$ in $(*)$ we finish the proof of the theorem. $\blacksquare$

Second example: representation theory of $GL(n,p)$

group of invertible $n \times n$ matrices over finite field $\mathbb{F}_p$

[field with p elements]

Irreducible representations of this group are discussed in Chapter IV of Macdonald's book and in:

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Representations of Finite Classical Groups

A Hopf Algebra Approach

Authors: **Zelevinsky**, A. V.

For a class of representations called unipotent, irreducibles are parameterized by partitions λ, $|\lambda|=n$

Let $\chi^\lambda(g)$, $g \in GL(n,q)$, be the character of such representation.

$\chi^\lambda(g)$ is central: $\chi^\lambda(aga^{-1}) = \chi^\lambda(g)$, i.e., this is a function of conjugacy class of g.

What are conjugacy classes in $GL(n,p)$?

Similarly to real and complex cases, one needs to study decomposition of the characteristic polynomial of a matrix into irreducible polynomial factors. Difference: over $\mathbb{F}_p$ there are many irreducible polynomials of every degree [While over $\mathbb{C}$ all of them are degree 1 and over $\mathbb{R}$ all of them are of degrees 1 or 2]

For simplicity let us consider only matrices with characteristic polynomial $(x-1)^n$. All their eigenvalues are 1 and by conjugations such matrix can be brought to Jordan Normal form with blocks $k\begin{pmatrix} 1 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}_{k}$. Conjugacy class of such matrices are parameterized by sizes of blocks.

Conclusion: We are trying to understand the values of characters of irreducible unipotent representations parameterized by λ, $|\lambda|=n$, on matrices with all eigenvalues 1 and JNF parameterized by μ, $|\mu|=n$, giving the sizes of the blocks.

Theorem: Set $t=p^{-1}$. Then for g with JNF μ

$$\chi^{\lambda}(g) = p^{\sum_i (i-1)\mu_i} K_{\lambda}^{\mu}(t)$$

, given by decomposition

$$s_{\lambda} = \sum_{\mu} K_{\lambda}^{\mu}(t)\, P_{\mu}(\,\cdot\,;0,t)$$

Schur

Hall-Littlewood

Remark: This is reminiscent of the character table for symmetric group S_n, but with products of power sums replaced by Hall-Littlewood polynomials.