Math 740 **Jack polynomials and random matrices** Lecture 21

Historically, Jack polynomials first appeared and were studied because of them being useful in random matrix theory and computations of matrix integrals.

Notation 1: When we discuss matrices with real/complex/quaternion matrix elements we use $\theta = \frac{1}{2} / 1 / 2$, respectively, in Jack polynomials.

Notation 2: If A is $N \times N$ matrix with eigenvalues $a_1, \dots a_N$, then

$$J_\lambda(A) := J_\lambda(a_1, \dots, a_N; \theta)$$

Theorem: Let A and B be positive-definite $N \times N$ Hermitian real/complex/quaternion matrices and let $\mathcal{U}$ be uniformly random (= distributed by the Haar measure) element of orthogonal/unitary/symplectic group at $\theta = \frac{1}{2}/1/2$, respectively. Then for each λ:

$$\mathbb{E}_{\mathcal{U}} \frac{J_\lambda(A\,\mathcal{U} B \mathcal{U}^*)}{J_\lambda(1^N)} = \frac{J_\lambda(A)}{J_\lambda(1^N)} \cdot \frac{J_\lambda(B)}{J_\lambda(1^N)} \qquad (*)$$

means integral over $\mathcal{U}$

Remark 1: $A\mathcal{U}B\mathcal{U}^*$ has the same distribution of eigenvalues as $VAV^* \cdot \mathcal{U}B\mathcal{U}^*$ with independent $\mathcal{U}$ and V.

Indeed, eigenvalues are preserved under conjugations, while $AV^*\mathcal{U}BW^*V = AWBW^*$, where $W = V^*\mathcal{U}$ is again uniformly distributed

Interpretation: Description of product of two random matrices.
[very useful in multivariate statistics]

Remark 2: Eigenvalues of $AUBU^*$ are real because they coincide with those of a Hermitian matrix

$$A^{\frac{1}{2}} U B U^* A^{1/2}$$

We will not prove the theorem (see Macdonald, Chapter VII)

Some Keywords: This is a computation of (zonal) spherical functions for Gelfand pairs

quaternions

$$(GL(N;\mathbb{C}), U(N)), \quad (GL(N;\mathbb{R}), O(N)), \quad (GL(N;\mathbb{H}), Sp(2N))$$

$(*)$ is a functional relation, which is always satisfied by such spherical functions. If $\theta = 1$ and A, B are unitary,

$$\int_{U(N)} \frac{s_\lambda(AUBU^*)}{s_\lambda(1^N)} = \frac{s_\lambda(A)}{s_\lambda(1^N)} \cdot \frac{s_\lambda(B)}{s_\lambda(1^N)}$$

– functional relation, which holds for characters of irreducible representations of any finite/compact group.

Expectations of Jack polynomials also appear in an extension of Selberg integral

at $\lambda=(0)$

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Regular Article

The Selberg–Jack Symmetric Functions

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But received in 1987

Theorem 1: For each λ, $r\geq 0$, $s\geq 0$

$$\frac{1}{N!}\int_0^1\cdots\int_0^1 J_\lambda(x_1,\dots,x_N;\theta)\prod_{i<j}|x_i-x_j|^{2\theta}\prod_{i=1}^{N}x_i^{r-1}(1-x_i)^{s-1}\,dx_i$$

$$=\prod_{1\leq i<j\leq N}\frac{\Gamma(\lambda_i-\lambda_j+\theta(j-i+1))}{\Gamma(\lambda_i-\lambda_j+\theta(j-i))}\cdot\prod_{i=1}^{N}\frac{\Gamma(\lambda_i+r+\theta(N-i))\,\Gamma(s+\theta(N-i))}{\Gamma(\lambda_i+r+s+\theta(2N-i-1))}$$

Meaning: This gives an explicit computation of $\mathbb{E}\,J_\lambda(x_1,\dots x_N)$ for $(x_1,\dots x_N)$ distributed as $\frac{1}{Z_N}\prod_{i<j}|x_i-x_j|^{2\theta}\prod_{i=1}^{N}x_i^{r-1}(1-x_i)^{s-1}dx_i$

$\beta=2\theta$ Jacobi ensemble of random matrix theory

Expanding other functions in basis of Jack polynomials one can compute expectations of arbitrary symmetric observables

Proof of Theorem: We know that Selberg integral is $q\to 1$, $t=q^\theta$ limit of (see Lecture 19)

$$\sum_\mu P_\mu(x_1,\dots x_N)\, Q_\mu(y_1,\dots y_M) = \prod_{i=1}^{N}\prod_{j=1}^{M} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}$$

$x_i = t^{i-1}$

$y_i = t^{d+i-1}$

We can write using label-variable duality [Lec. 18 slide 6]

$$\sum_\mu P_\lambda(q^{\mu_i} t^{N-i},\ 1\le i\le N)\, P_\mu(1,t,\dots t^{N-1})\, Q_\mu(t^d,\dots,t^{d+M-1}) =$$

$$= P_\lambda(1,t,\dots,t^{N-1})\cdot \sum_\mu \frac{P_\lambda(q^{\mu_i} t^{N-i})}{P_\lambda(1,t,\dots t^{N-1})}\cdot P_\mu(1,\dots t^{N-1})\, Q_\mu(t^d,\dots t^{d+M-1}) =$$

$$= P_\lambda(1,\dots t^{N-1}) \sum_\mu \frac{P_\mu(q^{\lambda_i} t^{N-i})}{P_\mu(1,t,\dots,t^{N-1})}\, P_\mu(1,\dots,t^{N-1})\, Q_\mu(t^d,\dots,t^{d+M-1}) =$$

$$= P_\lambda(1,\dots t^{N-1}) \sum_\mu P_\mu(q^{\lambda_i} t^{N-i})\, Q_\mu(t^d,\dots,t^{d+M-1}) =$$

$$= P_\lambda(1,\dots t^{N-1}) \prod_{i=1}^{N}\prod_{j=1}^{M} \frac{(t\, q^{\lambda_i}\, t^{N-i+d+M-j}; q)_\infty}{(q^{\lambda_i}\, t^{N-i+d+M-j}; q)_\infty}$$

Now we set $q = \exp(-\varepsilon)$, $t = q^{\theta}$, $r_i = \exp(-\varepsilon y_i) = q^{y_i}$ as in Lecture 19, Slide 3 and send $\varepsilon \to 0$.

$$P_\lambda\left(q^{\mu_1} t^{N-1}, \dots, q^{\mu_N}; q, t\right) \xrightarrow[\varepsilon \to 0]{} J_\lambda\left(r_1, \dots r_N;\ \theta\right)$$

As in Lecture 19, left-hand side of the identity becomes

$$\sim \int\limits_{r_1 < \dots < r_N} \cdots \int J_\lambda(r_1, \dots r_N; \theta) \prod_{i<j} |r_i - r_j|^{2\theta} \prod_{i=1}^{N} r_i^{d\theta - 1} (1-r_i)^{\theta(M-N+1)-1}\, dr_i$$

and it remains to compute the $\varepsilon \to 0$ limit of the explicit right-hand side $\blacksquare$

Exercise: Fill in the remaining details.

Question: Why should we care about such weird integrals of weird distributions?

Setup: Take two rectangular matrices with i.i.d. standard Gaussian entries: X of size $N \times T$ and Y of size $K \times T$. Let $S(X)$ be a subspace of of T-dimensional space spanned by N rows of X. Let $S(Y)$ be a subspace spanned by K rows of Y. Let P_X, P_Y be orthogonal projectors on $S(X)$, $S(Y)$, respectively.

$P_X P_Y P_X$ is a Hermitian matrix of rank $\min(N, K)$

Its eigenvalues are (squared) sample canonical correlations between X and Y

Remark: $\dim S(X) = N$, $\dim S(Y) = K$. These are two uniformly random subspaces of such dimensions.

Theorem 2: Assume $N \le K$, $N+K \le T$. Then the law of N positive eigenvalues $r_1 < r_2 < \dots < r_N$ of $P_X P_Y P_X$ has density

$$\frac{1}{Z} \prod_{1 \le i < j \le N} (r_j - r_i)^{2\theta} \prod_{i=1}^{N} r_i^{\theta(K-N+1)-1} (1-r_i)^{\theta(T-N-K+1)-1} \qquad (*)$$

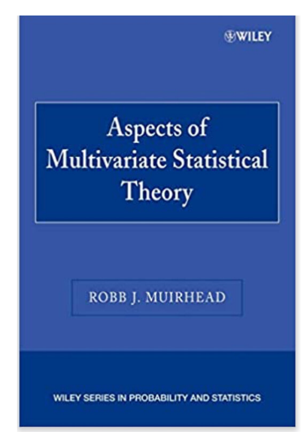
normalization consant

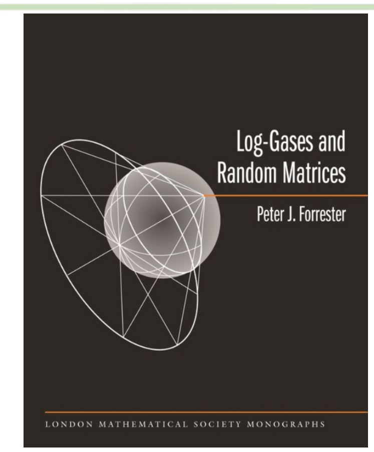
$\theta = \frac{1}{2} / 1 / 2$ corresponds to real/complex/quaternion matrix elements.

„Jacobi ensemble"

We will not give a proof, see:

[One needs to compute the Jacobian of transition to eigenvalues]





Example: Take $N = K = 1$. Then (Check this!) r_1 is the squared sample correlation coefficient

$$r_1 = \frac{\left(\sum x_i y_i\right)^2}{\left(\sum x_i^2\right)\left(\sum y_i^2\right)}$$

Exercise: Check that r_1 i Beta-distributed ($N = K = 1$ in $(*)$)

Conclusion: We can explicitly compute expectations of any symmetric polynomial in (squared) sample canonical correlations of two independent sets of data by expanding in Jack polynomials and using Kadell integrals.

Remark: (For those who know random matrices). Taking limits from Jacobi to Laguerre (= Wishart = sample covariance matrices) or Hermite (= Wigner = GOE/GUE/GSE) the same conclusion remains true for them

Example Characteristic polynomial of the matrix

$$\mathcal{X}(z) = \prod_{i=1}^{N} (z - r_i) = \sum_{m=0}^{N} e_m(r_1, \ldots r_N) \cdot z^{N-m}$$

Note that $e_m = J_{(1^m)}$ for each θ.

$$\mathbb{E}_{(\theta,\nu,s)\text{-Jacobi}}\, e_m(r_1, \ldots, r_N) = \frac{\text{Theorem 1 for } \lambda = (1^m)}{\text{Theorem 1 for } \lambda = (0)}$$

Substituting, we get:

$$\prod_{1\le i\le m<j\le N} \frac{\Gamma(1+\theta(j-i+1))}{\Gamma(\theta(j-i+1))} \cdot \frac{\Gamma(\theta(j-i))}{\Gamma(1+\theta(j-i))} \cdot \prod_{i=1}^{m} \frac{\Gamma(1+v+\theta(N-i))}{\Gamma(v+\theta(N-i))} \cdot \frac{\Gamma(v+s+\theta(2N-i-1))}{\Gamma(1+v+s+\theta(2N-i-1))}$$

We can simlify using $\Gamma(x+1) = x\,\Gamma(x)$

Unexpected outcome: Expected characteristic polynomial

$$\mathbb{E}_{(\theta,v,s)\text{-Jacobi}} \prod_{i=1}^{N} (z-r_i) = F_{v,s}(z)$$

does not depend on θ !

Same remains true for matrices from Slide 10 and slide 2

Identification of these polynomials with known objects goes beyond this class. For Jacobi example one gets Jacobi orthogonal polynomials (parameters are <u>not</u> v,s). For slide 2 one gets finite free (multiplicative) convolution of characteristic polynomials of A and B.