Math 740 Selberg integral Lecture 19

Theorem: [Selberg 1941–1944]

$$\int_0^1 \cdots \int_0^1 \prod_{i=1}^{N} t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le N} |t_i-t_j|^{2\theta}\, dt_1\ldots dt_N$$

$$= \prod_{j=0}^{N-1} \frac{\Gamma(\alpha+j\theta)\Gamma(\beta+j\theta)\Gamma(1+\theta+j\theta)}{\Gamma(\alpha+\beta+(N-1+j)\theta)\,\Gamma(1+\theta)}$$

This is a generalization of Beta integral at $N=1$:

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt = \frac{\Gamma(\alpha)(\beta)}{\Gamma(\alpha+\beta)}$$

The importance of the Selberg integral

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Nice and accessible historic overview

Interpretation: Theorem computes the normalization of Jacobi ensemble (distribution)
[Important in Random Matrix Theory]

Three approaches to the proof:

- Selberg: prove for $\theta = 1, 2, 3, ..$ by reductions to Beta-integrals. Then extend to $\theta > 0$ by complex analysis; need to check uniqueness of analytic continuation.

- Symmetric functions proof. (Details are coming!) This is a limit of (q,t)-Cauchy-Littlewood identity or orthogonality of Macdonald polynomials

- Direct inductive proof. The statement can be proven recursively in N, using Dixon-Anderson integration identity over domains of the form $x_1 \leq y_1 \leq x_2 \leq y_2 \leq ..$ [either x_i or y_i are fixed] These are closely related to branching rules.

Proof of Selberg's integral: We know that

$$(*) \quad \sum_{\lambda} P_\lambda(x_1, \dots, x_N; q, t)\, Q_\lambda(y_1, \dots y_M; q, t) = \prod_{i,j} \frac{(tx_i y_j; q)_\infty}{(x_i y_j; q)_\infty}$$

We set $x_i = t^{i-1}$, $1 \le i \le N$; $\quad y_i = t^{d+i-1}$, $1 \le i \le M$

Assume $M > N$. Then $\ell(\lambda) \le N$ in summation, that is, we sum over $\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_N$

Limit transition: $\varepsilon \to 0$

$$q = \exp(-\varepsilon), \quad t = q^\theta, \quad r_i = \exp(-\varepsilon \lambda_i)$$

Lemma: Suppose that as $q \to 1$, $u(q) \to u$, $0 < u < 1$.
Then for any a, b

$$\lim_{q \to 1} \frac{(q^a u(q); q)_\infty}{(q^b u(q); q)_\infty} = (1-u)^{b-a}$$

Exercise: Prove this by taking logarithm and aproximating sum $\approx$ integral.

By Lecture 18, slide 1. For $\lambda=(\lambda_1\geq\ldots\geq\lambda_N)$ and $M\geq N$

$$P_\lambda(1,\ldots,t^{M-1};q,t)=t^{\sum_{i=1}^N(i-1)\lambda_i}\prod_{i<j\leq M}\frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty}\cdot\frac{(t^{j-i+1};q)_\infty}{(t^{j-i};q)_\infty}$$

$$=t^{\sum_{i=1}^N(i-1)\lambda_i}\prod_{i<j\leq N}\frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty}\prod_{i=1}^{N}\prod_{j=N+1}^{M}\frac{(q^{\lambda_i}t^{j-i};q)_\infty}{(q^{\lambda_i}t^{j-i+1};q)_\infty}\prod_{i<j}^{i\leq N;\,j\leq M}\frac{(t^{j-i+1};q)_\infty}{(t^{j-i};q)_\infty}.$$

Using the last slide's Lemma:

$$t^{\sum_{i=1}^N\lambda_i(i-1)}\to\prod_{i=1}^N(\mathbf{r}_i)^{\theta(i-1)},\quad\frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty}\to\Big(1-\mathbf{r}_i/\mathbf{r}_j\Big)^\theta$$

$$\prod_{j=N+1}^{M}\frac{(q^{\lambda_i}t^{j-i};q)_\infty}{(q^{\lambda_i}t^{j-i+1};q)_\infty}=\frac{(q^{\lambda_i}t^{N+1-i};q)_\infty}{(q^{\lambda_i}t^{M+1-i};q)_\infty}\to\Big(1-\mathbf{r}_i\Big)^{\theta(M-N)},$$

And using Lemma from Lecture 18, slide 3: $\dfrac{(t^{j-i+1};q)_\infty}{(t^{j-i};q)_\infty}\sim\dfrac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i+1))}\varepsilon^{-\theta}$

By Lecture 18, Slide 5: $\dfrac{Q_\lambda(\cdot;q,t)}{P_\lambda(\cdot;q,t)}=b_\lambda=\displaystyle\prod_{1\leq i\leq j\leq\ell(\lambda)}\frac{f(q^{\lambda_i-\lambda_j}t^{j-i})}{f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})},\quad f(u)=\frac{(tu;q)_\infty}{(qu;q)_\infty}$

And using the same two Lemmas we get: $b_\lambda\sim\displaystyle\prod_{i=1}^N\frac{f(1)}{f(q^{\lambda_i}t^{N-i})}\sim\frac{\varepsilon^{N(1-\theta)}}{\Gamma(\theta)^N}\prod_{i=1}^N(1-\mathbf{r}_i)^{\theta-1}.$

Combining all the ingredients, the term in $(*)$ is:

combines nicely with this

$$P_\lambda Q_\lambda \sim \prod_{1\le i<j\le N}\left(1-\frac{r_i}{r_j}\right)^{2\theta}\prod_{i=1}^{N} r_i^{2\theta(i-1)}\, r_i^{\alpha\theta}\,(1-r_i)^{\theta(M-N)+\theta-1}$$

because $Q_\lambda(t^\alpha, t^{\alpha+1},\dots) = t^{\alpha|\lambda|}Q_\lambda(1,t,\dots)$

$$\prod_{1\le i<j\le N}\left[\frac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i+1))}\varepsilon^{-\theta}\right]^2\cdot\prod_{i=1}^{N}\prod_{j=N+1}^{M}\left[\frac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i+1))}\varepsilon^{-\theta}\right]\cdot\varepsilon^{N(1-\theta)}\,\Gamma(\theta)^{-N}$$

The sum over λ_i becomes an integral and we should take into account $dr_i \sim -\varepsilon r_i\, d\lambda_i$

For the right side of $(*)$ we again use Lemma from Lec18

$$\frac{(ta_ib_j;q)_\infty}{(a_ib_j;q)_\infty}=\frac{(t\cdot t^{i-1}\cdot t^{\alpha+j-1};q)_\infty}{(t^{i-1}\cdot t^{\alpha+j-1};q)_\infty}=f(t^{i-1}\cdot t^{\alpha+j-1})\sim\left[\frac{\Gamma(\theta(i-1+j-1+\alpha))}{\Gamma(\theta(i-1+j-1+\alpha+1))}\varepsilon^{-\theta}\right]$$

It remains to combine all factors and get:

$$\int \cdots \int_{0<r_1<\dots<r_N<1} \prod_{i<j} (r_j - r_i)^{2\theta} \prod_{i=1}^{N} r_i^{\alpha\theta - 1} (1-r_i)^{\theta(M-N+1)-1}$$

$$= (\text{product of bunch of } \Gamma\text{-functions})$$

Multiplying by $N!$ to adjust from $0<r_1<r_2<\dots<r_N<1$ to $t_i \in [0,1]$, we get the Selberg integral with α replaced by $\alpha\theta$ and β replaced by $\theta(M-N+1)$

[Note that $M-N+1 \in \mathbb{Z}_{>0}$ in our derivation — slightly less general]

Corollary (Mehta's integral)

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\le i<j\le N} |t_i - t_j|^{2\theta} \prod_{i=1}^{N} e^{-\frac{t_i^2}{2}} \, dt_1 \dots dt_N = (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1+j\theta)}{\Gamma(1+\theta)}$$

Exercise: Prove this by setting $\alpha = M-N+1 = L$ and sending $L \to \infty$ in Selberg [Compare with Lecture 9, slide 10]

Interpretation: Selberg integral is a particular case of Cauchy identity for a degeneration of Macdonald polynomi
[called Heckman–Opdam hypergeometric functions]

Proposition 6.4. *Let $M \geq N$ and suppose that*

$$t = q^\theta, \quad q = \exp(-\varepsilon), \quad \lambda = \lfloor h\varepsilon^{-1}(r_1, \dots, r_N, 0, \dots, 0)\rfloor, \quad x_i = \exp(\varepsilon y_i),$$

where $r_1 > r_2 > \cdots > r_N > 0$ and $\lambda \in \mathbb{GT}^+_M$. Then there exists a limit

$$\mathcal{F}_r(y_1, \dots, y_M; \theta) = \lim_{\varepsilon \to 0} \varepsilon^{\theta(N(N-1)/2 + N(M-N))} P_\lambda(x_1, \dots, x_M; q, t),$$

Proposition 6.5. *Let $M \geq N$ and suppose that*

$$t = q^\theta, \quad q = \exp(-\varepsilon), \quad \lambda = \lfloor \varepsilon^{-1}(r_1, \dots, r_N, 0, \dots, 0)\rfloor, \quad x_i = \exp(\varepsilon y_i),$$

where $r_1 > r_2 > \cdots > r_N > 0$ and λ is a signature of size M. Then there exists a limit

$$\widetilde{\mathcal{F}}_r(y_1, \dots, y_M; \theta, h) = \lim_{\varepsilon \to 0} \varepsilon^{N(\theta-1) + \theta(N(N-1)/2 + N(M-N))} Q_\lambda(x_1, \dots, x_M; q, t),$$

III. *Cauchy–type identity: Take N parameters $a_1, \dots, a_N$ and M parameters $b_1, \dots, b_M$ such that $a_i + b_j < 0$ for all i, j. Then*

$$\int_{r \in \widehat{\mathbb{GT}}_{\min(N,M)}} \widetilde{\mathcal{F}}_r(a_1, \dots, a_N; \theta) \mathcal{F}_r(b_1, \dots, b_M; \theta) \prod_{i=1}^{\min(N,M)} dr_i = \prod_{i,j} \frac{\Gamma(-a_i - b_j)}{\Gamma(\theta - a_i - b_j)}.$$

IV. *Principal specialization: Let $r \in \widehat{\mathbb{GT}}_N$ and $M \geq N$, then*

$$\mathcal{F}_r(0, -\theta, \dots, (1-M)\theta; \theta) = \prod_{i<j}^{i \leq N; j \leq M} \frac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i+1))} \prod_{i<j} (e^{-r_j} - e^{-r_i})^\theta \prod_{i=1}^N (1 - e^{-r_i})^{\theta(M-N)} \quad (87)$$

and

$$\widetilde{\mathcal{F}}_r(0, -\theta, \dots, (1-M)\theta; \theta) = \frac{1}{(\Gamma(\theta))^N} \prod_{i<j}^{i \leq N; j \leq M} \frac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i+1))} \prod_{i<j} (e^{-r_j} - e^{-r_i})^\theta \prod_{i=1}^N (1 - e^{-r_i})^{\theta(M-N) + (\theta-1)}. \quad (88)$$

Three together give Selberg integral

[Taken from

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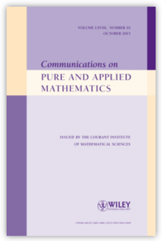
Alexei Borodin, Vadim Gorin



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]

Here is another face of Selberg's integral:

Theorem: constant term $\left[\prod_{1\le i<j\le N}\left(1-\frac{x_i}{x_j}\right)^{k}\left(1-\frac{x_j}{x_i}\right)^{k}\right]=\frac{(kN)!}{(k!)^N}$

- Conjectured by Dyson in 1962.

the first paper of future Nobel prize winner

- Proved by Gunson and Wilson in the same year

- Later Good found a one page proof of a more general statement

UW-Madison proof

- Askey explained that this is a particular case of Selberg integral

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Short Proof of a Conjecture by Dyson

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Dyson made a mathematical conjecture in his work on the distribution of energy levels in complex systems. A proof is given, which is much shorter than two that have been published before.

Let $G(\mathbf{a})$ denote the constant term in the expansion of

$$F(\mathbf{x};\mathbf{a})=\prod_{i\neq j}\left(1-\frac{x_j}{x_i}\right)^{a_j},\quad i,j=1,2,\cdots,n,$$

where $a_1, a_2, \cdots, a_n$ are nonnegative integers and where $F(\mathbf{x};\mathbf{a})$ is expanded in positive and negative powers of $x_1, x_2, \cdots, x_n$. Dyson[1] conjectured that $G(\mathbf{a})=M(\mathbf{a})$, where $M(\mathbf{a})$ is the multinomial coefficient $(a_1+\cdots+a_n)!/(a_1!\cdots a_n!)$. This was proved by Gunson[2] and by Wilson.[3] A much shorter proof is given here.

By applying Lagrange's interpolation formula (see, for example, Kopal[4]) to the function of x that is identically equal to 1 and then putting $x=0$, we see that

$$\sum_j\prod_i\left(1-\frac{x_j}{x_i}\right)^{-1}=1,\ i=j.$$

By multiplying $F(\mathbf{x};\mathbf{a})$ by this function we see that, if $a_j\neq 0$, $j=1,\cdots,n$, then

$$F(\mathbf{x};\mathbf{a})=\sum_j F(\mathbf{x};a_1,a_2,\cdots,a_{j-1},a_j-1,a_{j+1},\cdots,a_n),$$

so that

$$G(\mathbf{a})=\sum_j G(a_1,\cdots,a_{j-1},a_j-1,a_{j+1},\cdots,a_n).\tag{1}$$

If $a_j=0$, then x_j occurs only to negative powers in $F(\mathbf{x};\mathbf{a})$ so that $G(\mathbf{a})$ is then equal to the constant term in

$$F(x_1,\cdots,x_{j-1},x_{j+1},\cdots,x_n;a_1,\cdots,a_{j-1},a_{j+1},\cdots,a_n),$$

that is,

$$G(\mathbf{a})=G(a_1,\cdots,a_{j-1},a_{j+1},\cdots,a_n),\quad \text{if } a_j=0.\tag{2}$$

Also, of course,

$$G(\mathbf{0})=1.\tag{3}$$

Equations (1)–(3) clearly uniquely define $G(\mathbf{a})$ recursively. Moreover, they are satisfied by putting $G(\mathbf{a})=M(\mathbf{a})$. Therefore $G(\mathbf{a})=M(\mathbf{a})$, as conjectured by Dyson.

[1] F. J. Dyson, J. Math. Phys. **3**, 140, 157, 166 (1962).
[2] J. Gunson, J. Math. Phys. **3**, 752 (1962).
[3] K. G. Wilson, J. Math. Phys. **3**, 1040 (1962).
[4] Z. Kopal, *Numerical Analysis* (Chapman and Hall, London, 1955), p. 21.

Observation: The constant term is the same as

$$\frac{1}{(2\pi)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{a<b}\left|e^{i\varphi_a}-e^{i\varphi_b}\right|^{2K}d\varphi_1\cdots d\varphi_N=\frac{1}{(2\pi i)^N}\oint\cdots\oint_{|z_1|=\ldots=|z_N|=1}\prod_{i<j}\left((z_i-z_j)(z_i^{-1}-z_j^{-1})\right)^K\frac{dz_1}{z_1}\cdots\frac{dz_N}{z_N}$$

„Circular $\beta=2K$ ensemble"

Remark: For $K=1$ we already encountered this integral in Lecture 5 — this was the normalization constant of projection of the uniform (Haar) measure of $\mathcal{U}(N)$ onto eigenvalues of matrices

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Lemma 2: Take a Laurent polynomial $f(x_1,\ldots x_N)$. We have

$$\int_0^1\cdots\int_0^1(t_1\cdots t_N)^{\eta-1}f(t_1,\ldots t_N)\,dt_1\ldots dt_N=$$

$$=\frac{1}{(2\sin\pi\eta)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}e^{i\eta(\varphi_1+\ldots+\varphi_N)}f(-e^{i\varphi_1},\ldots,-e^{i\varphi_N})\,d\varphi_1\cdots d\varphi_N$$

for all η with large enough $\operatorname{Re}\eta$, so that LHS is well-defined.

Proof of Lemma: Take $f = x_1^{a_1}\cdot\ldots\cdot x_N^{a_N}$. Then LHS is

$$\int_0^1\cdots\int_0^1 t_1^{a_1+\eta-1}\cdot\ldots\cdot t_N^{a_N+\eta-1}\,dt_1\cdot\ldots\cdot dt_N = \frac{1}{a_1+\eta}\cdot\ldots\cdot\frac{1}{a_N+\eta}$$

and RHS is

$$\frac{(-1)^{a_1+\dots+a_N}}{(2\sin\pi\eta)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{i\varphi_1(\eta+a_1)}\cdot\ldots\cdot e^{i\varphi_N(\eta+a_N)}\,d\varphi_1\ldots d\varphi_N =$$

since $a_i\in\mathbb{Z}$

$$= \frac{(-1)^{a_1+\dots+a_N}}{(2\sin\pi\eta)^N}\cdot\prod_{j=1}^{N}\frac{e^{i\pi(\eta+a_j)}-e^{-i\pi(\eta+a_j)}}{i(\eta+a_j)} = \frac{1}{a_1+\eta}\cdot\ldots\cdot\frac{1}{a_N+\eta} \quad \blacksquare$$

Applying **Lemma 2**, we get

is a Laurent polynomial in $e^{i\varphi_a}$ for $k\in\mathbb{Z}_{\geq 0}$

$$\frac{1}{(2\pi)^N}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} e^{i\eta(\varphi_1+\dots+\varphi_N)}\prod_{i<j}\left|e^{i\varphi_i}-e^{i\varphi_j}\right|^{2k}d\varphi_1\ldots d\varphi_N =$$

$$= \left(\frac{\sin\pi\eta}{\pi}\right)^N\int_0^1\cdots\int_0^1\prod_{i<j}(t_i-t_j)^k(t_i^{-1}-t_j^{-1})^k\prod_{i=1}^{N}t_i^{\eta-1}\,dt_1\ldots dt_N =$$

$$= \left(\frac{\sin\pi\eta}{\pi}\right)^N\int_0^1\cdots\int_0^1\prod_{i<j}(t_i-t_j)^{2k}\prod_{i=1}^{N}t_i^{\eta-(N-1)k-1}\,dt_1\ldots dt_N\cdot(-1)^{\frac{N(N-1)}{2}}$$

This is Selberg integral at $\alpha = \eta - (N-1)\kappa$, $\beta = 1$, $\theta = \kappa$.

It remains to take the formula for the Selberg integral, notice that it is analytic in η and then send $\eta \to 0$ to get the desired result. $\blacksquare$