Math 740 Advanced theory of Macdonald polynomials Lecture 18

Most of the Schur functions developments except for the determinantal formulas have direct analogues for Macdonald polynomials.

We do not have time to cover all the proofs, [See Macdonald's book, Chapter VI], but we will detail the main statements.

Theorem 1 (Principal specialization)

$$P_\lambda(1,t,\ldots,t^{N-1};q,t) = t^{\sum_{i=1}^{N}(i-1)\lambda_i} \cdot \prod_{1\le i<j\le N} \frac{(q^{\lambda_i-\lambda_j}t^{j-i};q)_\infty}{(q^{\lambda_i-\lambda_j}t^{j-i+1};q)_\infty} \cdot \frac{(t^{j-i+1};q)_\infty}{(t^{j-i};q)_\infty}$$

[This is a rational function of (q,t).]

Examples for theorem 1:

- $q = t$ $\qquad s_\lambda(1, t, \ldots, t^{N-1}) = t^{\Sigma(i-1)\lambda_i} \cdot \prod_{i<j} \dfrac{1 - t^{(\lambda_i - i)-(\lambda_j - j)}}{1 - t^{j-i}}$

This is equivalent to Lecture 4, slide 10.

- $q = 0$ (Hall-Littlewood polynomials)

$$P_\lambda(1, \ldots, t^{N-1}; 0, t) = t^{\sum_i (i-1)\lambda_i} \cdot \prod_{i<j;\, \lambda_i \neq \lambda_j} \frac{1 - t^{j-i+1}}{1 - t^{j-i}}$$

$$= t^{\sum_i (i-1)\lambda_i} \cdot \prod_{i<j} \frac{1 - t^{j-i+1}}{1 - t^{j-i}} \cdot \prod_{i<j:\, \lambda_i = \lambda_j} \frac{1 - t^{j-i}}{1 - t^{j-i+1}} =$$

$$= t^{\sum_i (i-1)\lambda_i} \cdot \prod_{j=1}^{N} \frac{1 - t^{j}}{1 - t} \cdot \prod_{k=1}^{\infty} \prod_{j=1}^{m_k(\lambda)} \frac{1 - t}{1 - t^{j}}$$

$m_k(\lambda)$ ← # of parts k in λ

• $t=1$ Monomial symmetric functions [lecture 17, slide 10]

$$m_\lambda(1^N) = \frac{N!}{\prod_{k\geq 1} [m_k(\lambda)]!}$$

[sending $t \to 1$ in the previous case]

# of parts k in λ

Exercise: check this directly from definition of m_λ.

• Set $t = q^\theta$, send $q \to 1$. Jack symmetric polynomials

Lemma: $\lim_{q\to 1} (1-q)^{1-x} \frac{(q;q)_\infty}{(q^x;q)_\infty} = \Gamma(x)$

[q-Gamma function converges to Gamma function]
we do not give a proof

$$J_\lambda(1^N;\theta) = \prod_{i<j} \frac{\Gamma(\lambda_i - \lambda_j + \theta(j-i+1))}{\Gamma(\lambda_i - \lambda_j + \theta(j-i))} \cdot \frac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i+1))}$$

- Set $\theta = 1$ in the last formula to get s_λ again.
  Set $\theta = 0$ in the last formula to get m_λ again.

- $t = q^\theta$, $q = e^{-\varepsilon}$, $\lambda = (\lfloor \varepsilon^{-1} r_1 \rfloor, \dots, \lfloor \varepsilon^{-1} r_N \rfloor)$, $x_i = e^{\varepsilon y_i}$
  $\varepsilon \to 0$

Heckman-Opdam hypergeometric functions

$$\mathcal{F}_{r_1,\dots,r_N}(y_1,\dots,y_N;\theta) = \lim_{\varepsilon \to 0} \varepsilon^{\theta N(N-1)/2} P_\lambda(x_1,\dots,x_N;q,t)$$

$$\mathcal{F}_{r_1,\dots,r_N}(0,-\theta,\dots,(1-N)\theta;\theta) = \prod_{i<j} \frac{\Gamma(\theta(j-i))}{\Gamma(\theta(j-i-1))} \cdot \prod_{i<j} \left(e^{-r_i} - e^{-r_j}\right)^\theta$$

We do not yet know much about Hall-Littlewood, Jack, or Heckman-Opdam functions, but we see that formulas have such well-defined limits.

Theorem 2 : Define $f(u) = \dfrac{(tu;q)_\infty}{(qu;q)_\infty}$. We have

$$\frac{Q_\lambda(x_1,\dots x_N;q,t)}{P_\lambda(x_1,\dots x_N;q,t)} = \frac{1}{\langle P_\lambda, P_\lambda\rangle_{q,t}} = \prod_{1\le i\le j\le \ell(\lambda)} \frac{f(q^{\lambda_i-\lambda_j}t^{j-i})}{f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})}$$

- At $q=t$ $f(u)=1$ and $Q_\lambda = P_\lambda$.
- At $q=0$, $f(u) = 1-tu$
- $t=q^\theta$, $q\to 1$. $f(q^x) \sim (1-q)^{1-\theta} \dfrac{\Gamma(x+1)}{\Gamma(x+\theta)}$
- $t=1 \longrightarrow$ singular [as it should be, since scalar product explodes]

Theorem 3 (Label-variable symmetry)

Take any two Young diagrams λ, μ with $\ell(\lambda) \le N$, $\ell(\mu) \le N$

Then
$$\frac{P_\lambda(q^{\mu_1}t^{N-1}, q^{\mu_2}t^{N-2}, \dots, q^{\mu_N})}{P_\lambda(1, t, \dots, t^{N-1})} = \frac{P_\mu(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \dots, q^{\lambda_N})}{P_\mu(1, t, \dots, t^{N-1})}$$

- At $q=t$ this is a corollary of the determinantal formula

$$\frac{s_\lambda(q^{\mu_1+N-1}, \dots, q^{\mu_N})}{s_\lambda(1, q, \dots, q^{N-1})} = \prod_{i<j}(q^{N-i}-q^{N-j}) \frac{\det\left[q^{(\mu_i+N-i)(\lambda_j+N-j)}\right]_{i,j=1}^{N}}{\prod_{i<j}(q^{\lambda_i+N-i}-q^{\lambda_j+N-j})(q^{\mu_i+N-i}-q^{\mu_j+N-j})}$$

symmetric expression in λ, μ

- $t=q^\theta$, $q\to 1$

Heckman-Opdam function

Jack polynomials

$$\frac{J_\mu(e^{-r_1}, \dots, e^{-r_N}; \theta)}{J_\mu(1^N; \theta)} = \frac{\mathcal{F}_{r_1,\dots,r_N}(-\mu_1-\theta(N-1), \dots, -\mu_N; \theta)}{\mathcal{F}_{r_1,\dots,r_N}(-\theta(N-1), -\theta(N-2), \dots, 0; \theta)}$$

Theorem 4 (combinatorial formula; branching rule)

$$P_\lambda(x_1,\dots x_N;q,t) = \sum_{\mu \prec \lambda} x_N^{|\lambda|-|\mu|} \cdot \psi_{\lambda/\mu} \cdot P_\mu(x_1,\dots x_{N-1};q,t)$$

interlacing

$$\psi_{\lambda/\mu} = f(1)^{N-1} \prod_{1\le i<j<N} f(q^{\mu_i-\mu_j} t^{j-i}) \prod_{1\le i\le j<N} \frac{f(q^{\lambda_i-\lambda_{j+1}} t^{j-i})}{f(q^{\mu_i-\lambda_{j+1}} t^{j-i}) f(q^{\lambda_i-\mu_j} t^{j-i})}$$

$$f(u) = \frac{(tu;q)_\infty}{(qu;q)_\infty}$$

Should use $N=k$ in

Iterating we get

$$P_\lambda(x_1,\dots x_N;q,t) = \sum_{0=\lambda^{(0)}\prec\lambda^{(1)}\prec\lambda^{(2)}\prec\dots\prec\lambda^{(N)}=\lambda} \prod_{k=1}^{N} \left[x_i^{|\lambda^{(k)}|-|\lambda^{(k-1)}|} \psi_{\lambda^{(k)}/\lambda^{(k-1)}} \right]$$

- $q=t$: $\psi_{\lambda/\mu}=1$ and we get the result of Lecture 4, slide 6

Exercise: Setting q and t appropriately, obtain the combinatorical formulas for Jack, Hall-Littlewood, monomial, Heckman-Opdam.

Theorem 5 (Pieri rules) [Formulas from Macdonald, Chapter VI, section 6]

(6.24) *Let μ be a partition and r a positive integer. Then*

(i) $P_\mu g_r = \sum_\lambda \varphi_{\lambda/\mu} P_\lambda$,

(ii) $Q_\mu g_r = \sum_\lambda \psi_{\lambda/\mu} Q_\lambda$,

(iii) $Q_\mu e_r = \sum_\lambda \varphi'_{\lambda/\mu} Q_\lambda$,

(iv) $P_\mu e_r = \sum_\lambda \psi'_{\lambda/\mu} P_\lambda$.

In (i) and (ii) (resp. (iii) and (iv)) the sum is over partitions λ such that $\lambda - \mu$ is a horizontal (resp. vertical) r-strip, and the coefficients are given by

$$\varphi_{\lambda/\mu} = \prod_{1 \leqslant i \leqslant j \leqslant l(\lambda)} \frac{f(q^{\lambda_i - \lambda_j} t^{j-i}) f(q^{\mu_i - \mu_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})}$$

$$\psi_{\lambda/\mu} = \prod_{1 \leqslant i \leqslant j \leqslant l(\mu)} \frac{f(q^{\mu_i - \mu_j} t^{j-i}) f(q^{\lambda_i - \lambda_{j+1}} t^{j-i})}{f(q^{\lambda_i - \mu_j} t^{j-i}) f(q^{\mu_i - \lambda_{j+1}} t^{j-i})}.$$

Slightly different form of the same $\psi_{\lambda/\mu}$ from the last slide

$$\varphi'_{\lambda/\mu}(q,t) = \varphi_{\lambda'/\mu'}(t,q), \qquad \psi'_{\lambda/\mu}(q,t) = \psi_{\lambda'/\mu'}(t,q).$$

$$f(u) = \frac{(tu;q)_\infty}{(qu;q)_\infty}$$

- At $q=t$ $\psi = \varphi = \psi' = \varphi' = 1$ and we get HW2, Problems 1-2.

Exercise: Setting q and t obtain Pieri's formulas for Jack, Hall-Littlewood, monomial and Heckman-Opdam cases.

There are many more structures. For instance:

- Skew functions $P_{\lambda/\mu}$, $Q_{\lambda/\mu}$
- q-integral representation

(Shifted) Macdonald polynomials: q-Integral representation and combinatorial formula

Andrei Okounkov

Compositio Mathematica **112**, 147–182(1998) | Cite this article

- Macdonald operators are a part of larger algebra

JOURNAL ARTICLE

Double Affine Hecke Algebras and Macdonald's Conjectures

Ivan Cherednik

Annals of Mathematics

Second Series, Vol. 141, No. 1 (Jan., 1995), pp. 191-216 (26 pages)

- Non-symmetric Macdonald polynomials

Acta Math.
Volume 175, Number 1 (1995), 75-121.

← Previous article | TOC | Next articl

Harmonic analysis for certain representations of graded Hecke algebras

Eric M. Opdam

Nonsymmetric Macdonald polynomials

Ivan Cherednik

International Mathematics Research Notices, Volume 1995, Issue 10, 1995, Pages 483–515,

I. G. MACDONALD

Affine Hecke algebras and orthogonal polynomials

Astérisque, tome 237 (1996), Séminaire Bourbaki, exp. nº 797, p. 189-207

- Other combinatorial formulas

A combinatorial formula for Macdonald polynomials

Authors: J. Haglund, M. Haiman and N. Loehr
Journal: J. Amer. Math. Soc. **18** (2005), 735-761

Mathematics > Combinatorics

[Submitted on 2 Nov 2018 (v1), last revised 13 Feb 2019 (this version, v3)]

From multiline queues to Macdonald polynomials via the exclusion process

Sylvie Corteel, Olya Mandelshtam, Lauren Williams

ELSEVIER Advances in Mathematics

Volume 226, Issue 1, 15 January 2011, Pages 309-331

A combinatorial formula for Macdonald polynomials

Arun Ram [a, b], Martha Yip [b]

Matrix product formula for Macdonald polynomials

Luigi Cantini[1], Jan de Gier[3,2] and Michael Wheeler[2]

Published 24 August 2015 • © 2015 IOP Publishing Ltd

Journal of Physics A: Mathematical and Theoretical, Volume 48, Number 38

Mathematical Physics

[Submitted on 15 Apr 2019]

Nonsymmetric Macdonald polynomials via integrable vertex models

Alexei Borodin, Michael Wheeler

- Attempts on (q,t)-Littlewood-Richardson rule

A Littlewood–Richardson rule for Macdonald polynomials

Martha Yip

Mathematische Zeitschrift **272**, 1259–1290(2012) | Cite this article

Mathematical Research Letters **1**, 279–296 (1994)

MACDONALD'S POLYNOMIALS AND REPRESENTATIONS OF QUANTUM GROUPS

PAVEL I. ETINGOF AND ALEXANDER A. KIRILLOV, JR.

JOURNAL ARTICLE

Hilbert Schemes, Polygraphs and the MacDonald Positivity Conjecture

Mark Haiman

Journal of the American Mathematical Society

Vol. 14, No. 4 (Oct., 2001), pp. 941-1006 (66 pages)

- Representation-theoretic realizations

But we proceed to applications