Math 740 Macdonald polynomials in Λ Lecture 17

So far: $P_\lambda(x_1, \ldots x_N; q, t) \in \Lambda_N$ through D_K^N or $\langle \cdot, \cdot \rangle_{q,t}$

How to define $N=\infty$ version of P_λ?

Difficulty: D_K^N are <u>not</u> consistent with projections $\Lambda_N \to \Lambda_{N-1}$

Indeed, for the same λ, eigenvalues are different:

$$D_1^N P_\lambda = \sum_{i=1}^{N} q^{\lambda_i} t^{N-i} P_\lambda$$

$$D_1^{N-1} P_\lambda = \sum_{i=1}^{N-1} q^{\lambda_i} t^{N-i-1} P_\lambda$$

Solution: we need to correct for this discrepancy

It can be done for each D_K^N, but we only do for D_1^N

Definition: $\widetilde{D}^N = t^{-N} D_1^N - \sum_{i=1}^{N} t^{-i}$

Proposition: Operators $\widetilde{D}^N$ commute with projections $\rho_{N,N-1} : \Lambda_N \to \Lambda_{N-1}$ setting $x_N = 0$:

$$\rho_{N,N-1} \circ \widetilde{D}^N = \widetilde{D}^{N-1} \circ \rho_{N,N-1}$$

Proof:

$$\mathrm{LHS}(f) = \rho_{N,N-1} \circ \left[t^{-N} \sum_{i=1}^{N} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} T_{q,i} - \sum_{i=1}^{N} t^{-i} \right] f(x_1, \dots x_N)$$

$$= \sum_{i=1}^{N-1} t^{1-N} \prod_{\substack{j \neq i \\ 1 \le j < N}} \frac{t x_i - x_j}{x_i - x_j} T_{q,i}\, f(x_1, \dots, x_{N-1}, 0) + t^{-N} \cdot f(x_1, \dots, x_{N-1}, 0)$$

cancels i=N term

$$- \sum_{i=1}^{N} t^{-i} f(x_1, \dots, x_{N-1}, 0) = \widetilde{D}^{N-1} f(x_1, \dots x_{N-1}, 0) =$$

$$= \mathrm{RHS}(f). \quad \blacksquare$$

Exercise: Give another proof of proposition using the formula for $D_1^N m_\lambda$ from Lecture 15

Theorem: For each λ with $\ell(\lambda) \le N$

$$P_\lambda(x_1, \dots, x_{N-1}, 0; q, t) = \begin{cases} P_\lambda(x_1, \dots x_{N-1}; q, t), & \ell(\lambda) \le N-1, \\ 0, & \text{otherwise}. \end{cases}$$

Proof: Let us start from the „otherwise" part

We know that $P_\lambda = m_\lambda + \sum_{\mu < \lambda} (\dots)\, m_\mu$ $\quad (*)$

If $\ell(\lambda) > N-1$, then also $\ell(\mu) > N-1$ for each $\mu < \lambda$. Hence, when we plug $x_N = 0$ into $(*)$, all terms vanish.

Next, assume $\ell(\lambda) \le N-1$. Then by using Proposition [last slide]

$$\widetilde{D}^{N-1} \rho_{N,N-1} P_\lambda = \rho_{N,N-1} \widetilde{D}^N P_\lambda = \rho_{N,N-1} \overbrace{\left(\sum_{i=1}^N (q^{\lambda_i} - 1) t^{-i} \right)}^{\text{eigenvalue of } \widetilde{D}^N} P_\lambda =$$

$$= \left(\sum_{i=1}^{N-1} (q^{\lambda_i} - 1) t^{-i} \right) \rho_{N,N-1} P_\lambda$$

the $i=N$ term is 0

Thus, $\rho_{N,N-1} P_\lambda$ is an eigenvector of $\widehat{D}^{N-1}$ with eigenvalue $\sum_i (q^{\lambda_i}-1)t^{-i}$. Thus, it is $P_\lambda(x_1, \dots, x_{N-1}; q, t)$ ■

Definition $P_\lambda \in \Lambda$ is limit of $P_\lambda(x_1, \dots, x_N; q, t)$, i.e. it is the sequence $\left(\underset{\Lambda_1}{\underset{\uparrow}{0}}, 0, \dots, \underset{\Lambda_{N-1}}{\underset{\uparrow}{0}}, P_\lambda(x_1, \dots x_N; q, t), \underset{\Lambda_{N+1}}{\underset{\uparrow}{P_\lambda(x_1, \dots x_{N+1}; q, t)}}, \dots\right)$

$[N = \ell(\lambda)]$

Definition: $\widehat{D} = \lim_{N\to\infty} \widehat{D}^N$: it acts on sequences

$$\widehat{D}\left(\underset{\Lambda_1}{\underset{\uparrow}{f_1}}, f_2, \underset{\Lambda_3}{\underset{\uparrow}{f_3}}, \dots\right) = \left(\widehat{D}^1 f_1, \widehat{D}^2 f_2, \widehat{D}^3 f_3, \dots\right)$$

Theorem: $P_\lambda \in \Lambda$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_{q,t}$:
$\langle P_\lambda, P_\mu \rangle = 0$ unless $\lambda = \mu$. They satisfy $\widehat{D} P_\lambda = \sum_{i=1}^{\infty} (q^{\lambda_i} - 1) t^{-i} P_\lambda$

the sum is actually finite

Proof: $N \to \infty$ in finite N results ■

Question: How should one think about $\widetilde{D}$?
Each $\widetilde{D}^N$ was a difference operator. But what does it mean to be a difference operator in Λ???

Identify $\Lambda = \mathbb{R}[p_1, p_2, \ldots]$. In this representation, we have operators p_k – multiplication by p_k

$\frac{\partial}{\partial p_k}$ – partial derivative by p_k

[This is like $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$, but now the variables are called p_k]

Theorem: Set $\eta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^{-n}}{n} z^n p_n\right) \exp\left(-\sum_{n=1}^{\infty} (1-q^n) z^{-n} \frac{\partial}{\partial p_n}\right)$

Then $$\widehat{D} = \frac{1 - [z^0]\eta(z)}{1-t}$$

constant term η_0 in expansion $\eta(z) = \sum_{k \in \mathbb{Z}} \eta_k z^k$

Some Keywords: Heisenberg representation, Fock space, vertex operators, Shuffle algebra

Sketch of the proof of theorem:

I) It suffices to check the identity on products $\prod_{i=1}^{n} f(x_i)$ [their linear combinations span everything]

II) $\exp\left(c\frac{\partial}{\partial p_k}\right)$ shifts p_k : $p_k \to p_k + c$

III) $[z^0]\eta(z) = \frac{1}{2\pi i}\oint_0 \frac{\eta(z)}{z}\,dz$ [integral around zero ⓞ]

IV) HW 4, Problem 3 gives a contour integral representation for D_1^N, which, however, was not applicable to $x_i = 0$, while we need that for $N\to\infty$. One should deform contour through 0, picking the residue there.

V) Eventually $\boxed{\text{III}} = \boxed{\text{IV}}$

Exercise: Fill in the details of this argument.

[This takes efforts!]

Conclusion: $\hat{D}$ is a differential operator in Λ, given by infinite sum of $p_i, \frac{\partial}{\partial p_j}$ products.

Another important player in our study of Λ was involution ω. How does it work with (q,t)? We need to adjust the definition.

Definition: $\omega_{q,t} : \Lambda \to \Lambda$ homomorphism given on generators by $p_k \to (-1)^{k-1} \dfrac{1-q^k}{1-t^k} p_k$

Important: It is no longer an involution!

Instead $\omega_{q,t} \circ \omega_{t,q} = \mathrm{Id}$

Proposition: $w_{q,t}\left(g_n(\cdot\,;q,t)\right) = e_n \qquad (*)$

Proof: $\sum_{n\geq 0} g_n z^n = \exp\left(\sum_{n\geq 1} \frac{1}{n}\frac{1-t^n}{1-q^n} p_n \cdot z^n\right)$

(see Lecture 16, slide 3)

Hence, $\sum w_{q,t}(g_n)\, z^n = \exp\left(\sum \frac{1}{n}(-1)^{n-1} p_n z^n\right) =$

$= E(z) = \sum_{n\geq 0} e_n z^n \qquad \blacksquare$

Corollary: $w_{q,t}(e_n) = g_n(\cdot\,;t,q)$

Proof: Apply $w_{t,q}$ to $(*)$ and swap $q \leftrightarrow t$ $\blacksquare$

What about the action on Macdonald polynomials?

Definition: $Q_\lambda(\cdot\,; q,t) = P_\lambda(\cdot\,; q,t) \cdot \underbrace{\langle P_\lambda, P_\lambda \rangle_{q,t}^{-1}}_{\text{will be computed later}}$

Proposition: (q,t) Cauchy identity

$$\sum_{\lambda \in \mathbb{Y}} P_\lambda(x_1, x_2, \dots)\, Q_\lambda(y_1, y_2, \dots) = \prod_{i,j=1}^{\infty} \frac{(t x_i y_j; q)_\infty}{(x_i y_j; q)_\infty}$$

Proof: $\langle P_\lambda, Q_\mu \rangle_{q,t} = \delta_{\lambda=\mu}$, which implies the statement by Lecture 16, Slide 4. $\square$

Theorem: $\omega_{q,t}(P_\lambda(\cdot\,; q,t)) = Q_{\lambda'}(\cdot\,; \underline{\underline{t,q}})$

We will not prove this, see Macdonald's book Chapter VI, Section 5.

Some further properties:

- $P_\lambda(\cdot\,;q,t) = P_\lambda(\cdot\,;q^{-1},t^{-1})$ because

degree

$\langle f, g\rangle_{q^{-1},t^{-1}} = (q^{-1}t)^n \langle f, g\rangle_{q,t}$ for $f,g \in \Lambda^n$

[by expanding f,g in power sums and using Lecture 16, slide 4]

Hence, orthogonalization procedures in $\langle\cdot,\cdot\rangle_{q^{-1},t^{-1}}$ and in $\langle\cdot,\cdot\rangle_{q,t}$ are the same

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- $P_\lambda(\cdot\,;q,1) = m_\lambda$

$D_1^N = \sum_{i=1}^{N} T_{q,i} \Rightarrow \underbrace{D_1^N m_\lambda = \sum_{i=1}^{N} q^{\lambda_i} \cdot m_\lambda}_{\text{so that's the desired eigenfunction}}$

- $P_\lambda(\cdot\,;q,q) = s_\lambda$, because this is true

in finite variable case, comparing Lec. 15, slide 6 with Lec 14, slide 6