Math 740

Orthogonality of Macdonald polynomials

Lecture 16

Last time: $P_\lambda(x_1, .. x_n; q, t)$ as eigenfunctions of D_K^N

Today: we find a scalar product, which makes D_K^N symmetric and P_λ orthogonal.

Reminder: For Schur functions we did this in Lecture 5 for N variables and in Lecture 8 for $N=\infty$.

These were two faces of the same scalar product.

Which is no longer true for general (q,t) case.

We start by producing a (q,t) version of $N=\infty$ case.

In Schur case we defined $\langle \cdot, \cdot \rangle$ by specifying

$\langle p_\lambda, p_\mu \rangle = (..)$ or $\langle h_\lambda, m_\mu \rangle = (..)$

We will try to repeat this approach

Definition: (q-Γamma function / q-Pochammer symbol)

$$(a;q)_\infty = \prod_{i=1}^{\infty} (1-aq^{i-1}) \qquad (a;q)_n = \prod_{i=1}^{n} (1-aq^{i-1})$$

$$(a;q)_1 = (1-a) \qquad (a;q)_2 = (1-a)(1-aq)$$

Definition: Symmetric functions g_n are found from the expansion

$$\prod_{i\geq 1} \frac{(tx_i z;q)_\infty}{(x_i z;q)_\infty} = \sum_{n=0}^{\infty} g_n(x_1,x_2,\dots;q,t)\, z^n$$

Example: When $q=t$, $\dfrac{(tx_i z;q)_\infty}{(x_i z;q)_\infty} = \dfrac{1}{1-x_i z}$ check this!

Hence, comparing with $H(z)$, we conclude $g_n(\cdot\,;q,q) = h_n$

g_n are (q,t)-deformations of complete homogeneous functions

Theorem: Define $g_\lambda = \prod_{i\geq 1} g_{\lambda_i}$. Then

$$\begin{tikzcd}
\sum_{\lambda\in\mathbb{Y}} g_\lambda(x_1,\dots)\, m_\lambda(y_1,\dots) \arrow[r, equal] \arrow[d, equal, "(*)"'] & \sum_\lambda \frac{p_\lambda(x_1,\dots)\, p_\lambda(y_1,\dots)}{z_\lambda(q,t)} \\
\prod_{i,j\geq 1} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} \arrow[ur, equal, "(**)"'] &
\end{tikzcd}$$

where $z_\lambda(q,t) = \underset{\uparrow}{z_\lambda} \prod_{i=1}^{\ell(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$

$\prod_i i^{m_i}(m_i)!$ as in Lecture 8

Proof: The identity $(*)$ is essentially the definition of g_n

[Exercise: $F(y) = 1 + \sum_{k\geq 1} c_k y^k \iff \prod_{j\geq 1} F(y_j) = \sum_\lambda c_\lambda m_\lambda(y_1, y_2, \dots)$]

For $(**)$: $\ln\left(\prod_{i,j} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}\right) = \sum_{i,j}\sum_{r\geq 0}\left[\ln(1-q^r t x_iy_j) - \ln(1-q^r x_iy_j)\right]$

$$= \sum_{i,j}\sum_{r\geq 0}\sum_{n\geq 1} \frac{1}{n}(1-t^n)(q^r x_iy_j)^n \underset{\text{sum over } r}{=} \sum_{i,j}\sum_{n\geq 1}\frac{1}{n}\frac{1-t^n}{1-q^n}(x_iy_j)^n = \sum_{n\geq 1}\frac{1}{n}\frac{1-t^n}{1-q^n}p_n(x_1,\dots)\,p_n(y_1,\dots)$$

Exponentiating the last identity, we get (**)
[as in Lecture 8, slide 2]

▨

Definition: $\langle \cdot , \cdot \rangle_{q,t}$ is a scalar product on Λ defined by either of 3 equivalent identities:

- $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda=\mu} \; z_\lambda(q,t)$
- $\langle m_\lambda, g_\mu \rangle = \delta_{\lambda=\mu}$
- $\langle v_\lambda , u_\mu \rangle = \delta_{\lambda=\mu}$ for any systems v_λ, u_λ, satisfying $$\sum_\lambda v_\lambda(x_1,\dots)\, u_\lambda(y_1,\dots) = \prod_{i,j\geq 1} \frac{(t x_i y_j ; q)_\infty}{(x_i y_j ; q)_\infty}$$

We silently assume here that (q,t) are generic, so that $z_\lambda(q,t)$ does not vanish or explode.

Equivalence is proven in the same way as for $q=t$ case [Lecture 8, slides 3-5]

Corollary: $g_n, n \geq 1$ are algebraically independent and $\{g_\lambda\}_{\lambda \in \mathbb{Y}}$ form a linear basis of Λ

Proof: $g_\lambda, \lambda \in \mathbb{Y}_n$ is a system of $\dim(\Lambda^n)$ degree n polynomials elements of Λ^n which are biorthogonal with respect to non-degenerate scalar product $\langle \cdot , \cdot \rangle_{q,t}$ to a linear basis $m_\lambda, \lambda \in \mathbb{Y}_n$ in Λ^n. Hence, $g_\lambda, \lambda \in \mathbb{Y}_n$ is a linear basis in Λ^n ∎

What about finitely many variables, i.e. Λ_N?

Proposition: $\{g_\lambda(x_1, \dots, x_N)\}_{\lambda \in \mathbb{Y}, \ell(\lambda) \leq N}$ form a linear basis of Λ_N.
[for generic (q,t)]

Remark: $g_1, g_2, g_3, \dots$ are not algebraic generators of Λ_N
[there are too many of them]

Proof of Proposition: Since g_λ are homogeneous of degree $|\lambda|$ and $\{m_\lambda\}_{\ell(\lambda)\leq N}$ form a basis of Λ_N, we can expand

$$g_\lambda = \sum_{\substack{|\mu|=|\lambda| \\ \ell(\mu)\leq N}} C_\lambda^\mu \, m_\mu$$

Fix $n=1,2,..$ and consider the matrix

$$A_n = [\, C_\lambda^\mu \,]_{|\lambda|=|\mu|=n\,,\ \ell(\lambda)\leq N,\ \ell(\mu)\leq N}$$

If we manage to prove that A_n is non-degenerate, then we are done. (why?)

This would follow from two facts:

I) $\det(A_n)$ is a rational function of (q,t)
[ratio of two polynomials in (q,t)]

II) At $q=t$, $\det(A_n)\neq 0$.

Indeed, then $\det A_n$ is not identical 0 as a rational function of (q,t) $\Rightarrow$
$\Rightarrow$ $\det A_n \neq 0$ for generic q,t $\Rightarrow$ A_n is non-degenerate for generic q,t.

For I notice that $g_m = \sum_{|\lambda|=m} z_\lambda^{-1}(q,t)\, p_\lambda$ through theorem of slide 3 with $y_2 = y_3 = .. = 0$. Restricting to N variables and expanding products of p_λ in m_μ, we see that all matrix elements of A_n are rational functions and so is its determinant.

For II recall that $g_m = h_m$ in this case Jacobi-Trudi formula implies that the transition matrix between $\{s_\lambda\}_{\ell(\lambda)\leq N}$ and $\{h_\lambda\}_{\ell(\lambda)\leq N}$ is unitriangular (with respet to dominance order) and so is the transition matrix between $\{s_\lambda\}_{\ell(\lambda)\leq N}$ and $\{m_\lambda\}_{\ell(\lambda)\leq N}$. Hence, $\det A_n = 1$ in this case by triangularity. $\blacksquare$

Definition: Scalar product $\langle \cdot , \cdot \rangle_{q,t}$ on Λ_N is defined by either of the equivalent definitions:

- $\langle g_\lambda , m_\mu \rangle = \delta_{\lambda=\mu}$ $\qquad \ell(\lambda) \le N, \ell(\mu) \le N$
- For any two bases $\{v_\lambda\}$, $\{u_\mu\}$, $\ell(\lambda)\le N$, $\ell(\mu)\le N$ satisfying $\sum_{\ell(\lambda)\le N} v_\lambda(x_1,\dots)\, u_\lambda(y_1,\dots) = \prod_{i,j=1}^{N} \frac{(t x_i y_j ; q)_\infty}{(x_i y_j ; q)_\infty}$ we have $\langle v_\lambda , u_\mu \rangle = \delta_{\lambda=\mu}$.

Equivalence — same argument as for $N=\infty$ version

Theorem: Macdonald difference operators D_N^k are symmetric with respect to $\langle \cdot , \cdot \rangle_{q,t}$.

This means: $\forall f, g \in \Lambda_N$, $\quad \forall k = 1, 2, \dots$

$$\langle D_N^k f , g \rangle_{q,t} = \langle f , D_N^k g \rangle_{q,t}$$

Proof. **Step 1:**

We show that

$$\underset{\uparrow}{D^N_{K;x}} \prod_{i,j=1}^{N} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty} = \underset{\uparrow}{D^N_{K;y}} \prod_{i,j=1}^{N} \frac{(tx_iy_j;q)_\infty}{(x_iy_j;q)_\infty}$$

acts in x-variables — acts in y-variables

Denote $\boldsymbol{\Pi} = \prod_{i,j=1}^{N} \frac{(tx_iy_j;q)_\infty}{(x_i,y_j;q)_\infty}$

Notice that $\boldsymbol{\Pi}^{-1} T_{q,x_i} \boldsymbol{\Pi} = \prod_{j=1}^{N} \frac{1-x_iy_j}{1-tx_iy_j}$, which **miraculously** does not depend on q!

Hence, both $\boldsymbol{\Pi}^{-1} D^N_{K;x} \boldsymbol{\Pi}$ and $\boldsymbol{\Pi}^{-1} D^N_{K;y} \boldsymbol{\Pi}$ are also q-indendent and it suffices to prove the identity at $q=t$. In this situation we use Cauchy-Littlewood:

$$\boldsymbol{\Pi} = \sum_{\lambda:\ell(\lambda)\le N} S_\lambda(x_1,..x_N)\, S_\lambda(y_1,..,y_N)$$

By eigenrelation [Lecture 14 slides 6-7]: $D^N_{K;x}\boldsymbol{\Pi} = \sum_{\ell(\lambda)\le N} e_K(q^{\lambda_i+N-i}) S_\lambda(x_1,..) S_\lambda(y_1,..) = D^N_{K;y}\boldsymbol{\Pi}$

Step 2 Take any orthonormal $\overset{\text{homogeneous}}{\downarrow}$ basis V_λ : $\langle V_\lambda, V_\mu \rangle_{q,t} = \delta_{\lambda=\mu}$.

Then by slide 8 we have $\sum\limits_{\ell(\lambda)\le N} V_\lambda(x_1,..)\, V_\lambda(y_1,..) = \Pi$

Set M to be the matrix of D_K^N in basis $\{V_\lambda\}$

$$D_K^N \; V_\lambda = \sum_\mu M_\lambda^\mu \; V_\mu$$

We want to show the symmetry $M_\lambda^\mu \overset{?}{=} M_\mu^\lambda$

We have $\sum\limits_{\lambda,\mu} M_\lambda^\mu V_\mu(x_1,..) V_\lambda(y_1,..) = D^N_{K;x} \, \Pi$

$$\sum_{\lambda,\mu} M_\mu^\lambda \; V_\mu(x_1,..) V_\lambda(y_1,..) = \overset{\|}{D^N_{K;y}} \; \Pi$$

Comparing coefficient of $V_\mu(x_1,..)\, V_\lambda(y_1,..)$, we are done ■

Remark: q-independence of $\Pi^{-1} D_K^N \Pi$ was recently used in Integrable probability to reduce analysis of complicated objects (directed polymers, KPZ-equation) to much simpler ones (Last Passage Percolation, Schur measures)

Corollary 1: Eigenvectors P_λ of D_K^N are orthogonal:

$\langle P_\lambda, P_\mu \rangle_{q,t} = 0$ unless $\lambda = \mu$ [$\langle P_\lambda, P_\lambda \rangle_{q,t}$ still needs to be computed]

Proof: Eigenvectors of any symmetric operator are orthogonal. ☑

Corollary 2: Macdonald polynomials P_λ can be obtained from m_λ by Gramm-Schmidt $\langle \cdot , \cdot \rangle_{q,t}$ orthogonalization with respect to any linear order extending dominance.

Proof: $P_\lambda = m_\lambda + \sum_{\mu<\lambda} (\dots) m_\mu$ by Lecture 15, slide 11 and they are orthogonal by Corollary 1. ■

Corollary 3: Operators D_K^N commute and their eigenvectors P_λ do not depend on the choice of $K = 1, 2, \dots N$.

Proof: For each $K = 1, \dots N$ the eigenvectors are obtained by procedure of Corollary 2. ☑

Example 1: $P_{(1^k)}(x_1,\dots x_N;q,t) = e_k(x_1,\dots x_N)$ for $k \leq N$.

[no dependence on q, t !]

Indeed, $P_{(1^k)} = m_{1^k} + \sum_{\mu<1^k} (..)\, m_\mu$.

This is e_k there are no such μ

Example 2: $P_{(k,0,0..)}(x_1,..x_N;q,t) = \dfrac{(q;q)_k}{(t;q)_k}\, g_k(x_1,\dots,x_N;q,t)$

Indeed, g_k is a degree k polynomial, orthogonal to all m_μ, $|\mu|=k$, $\mu \neq (k,0,..)$ and, hence, (by triangularity) to all P_μ, $|\mu|=k$, $\mu\neq(k,0,..)$.
Since P_μ, $|\mu|=k$ form a basis of degree k polynomials, this implies $g_k = c_k \cdot P_{(k,0,..)}$. Comparing leading coefficients, we compute the constant c_k.

Exercise: $P_{\lambda+1^N}(x_1,\dots x_N;q,t) = (x_1\cdot x_2\cdot\dots\cdot x_N)\, P_\lambda(x_1,\dots x_N;q,t)$

add 1 to each row

Hint: Apply D_1^N to both sides.