Math 740 Macdonald polynomials Lecture 15

Plan: 1) Introduce a family of operators, depending on (q,t)

2) Define Macdonald polynomials $P_\lambda(\cdot\,; q,t)$ as their eigenfunctions

---

History:

Hall-Littlewood polynomials $P_\lambda(\cdot\,; 0,t)$

Schur polynomials at $t=0$

Monomial symmetric functions at $t=1$

Related to finite abelian p-groups with $t=p^{-1}$

Dudley E. Littlewood

Proceedings of the London Mathematical Society

LONDON MATHEMATICAL SOCIETY EST. 1865

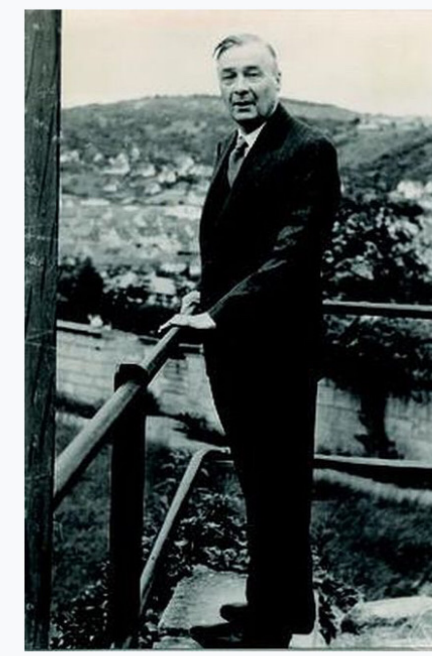
Articles

On Certain Symmetric Functions

D. E. Littlewood

First published: 1961 | https://doi.org/10.1112/plms/s3-11.1.485 | Citations: 35

Philip Hall



Philip Hall

Born	11 April 1904 Hampstead, London, England
Died	30 December 1982 (aged 78) Cambridge, England
Nationality	British
Alma mater	University of Cambridge
Known for	Hall's marriage theorem Hall polynomial Hall subgroup Hall–Littlewood polynomial

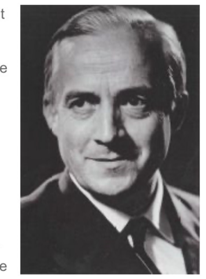
In 1950's

In parallel: Study of matrix integrals...

Professor Alan Treleven James (1924-2013)
Papers 1950-2013

MSS 0173

Biographical Note



Born in Berri, South Australia, in 1924, Alan Treleven James was the youngest child of litigant and dried fruit merchant, Frederick Alexander James and Rachel May James. Alan James attended the Glen Osmond Primary School and later won the Samuel Fiddian scholarship to attend Prince Alfred College. In 1944 he completed a Bachelor of Science with Honours at the University of Adelaide and a Masters of Science in 1949.

Alan James' first professional job with CSIR, now CSIRO, included teaching throughout his early twenties at The University of Adelaide. During a posting to CSIR in Canberra he met a colleague, Cynthia, who would become is wife in 1950. Shortly afterwards, Alan James was awarded a CSIRO studentship to study at Princeton University, New Jersey, where he completed his PhD in 1952. Alan James and Cynthia returned to Adelaide and their first two children, Michael and Stephen, were born.

Alan James continued to work for CSIRO until 1958, when he resigned to take up a one year teaching position at Yale University in Connecticut. He progressed to full professorship and he and Cynthia had two more sons, Andrew and Nicholas.

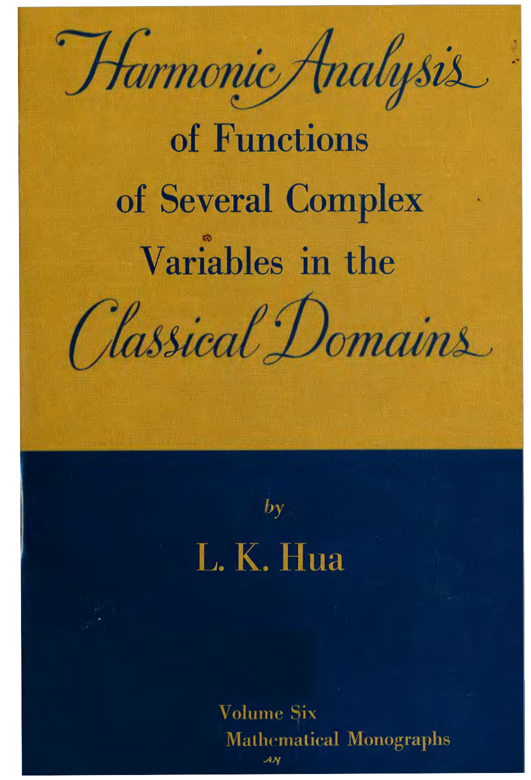
In 1965, Alan James and his family returned to Adelaide again, where he took on the role of Chair of Mathematical

JOURNAL ARTICLE

Zonal Polynomials of the Real Positive Definite Symmetric Matrices

Alan T. James

Annals of Mathematics
Second Series, Vol. 74, No. 3 (Nov., 1961), pp. 456-469 (14 pages)



Hua
Luogeng

Chinese mathematician

Hua Luogeng or Hua Loo-Keng was a Chinese mathematician and politician famous for his important contributions to number theory and for his role as the leader of mathematics research and education in the People's Republic of China. Wikipedia

Born: November 12, 1910, Jintan District, Changzhou, China

Died: June 12, 1985, Tokyo, Japan

... has led to introduction of Jack symmetric polynomials $J_\lambda(\cdot\,;\theta)$

Henry Jack

Quick Info

Born
6 July 1917
Menzieshill, near Dundee, Scotland

Died
5 January 1978
Dundee, Scotland

Proceedings of the Royal Society of Edinburgh Section A: Mathematics

Article Metrics

Volume 69, Issue 1 1970 , pp. 1-18

I.—A class of symmetric polynomials with a parameter*

Henry Jack (a1)

$\theta = \frac{1}{2}, 1, 2$ corresponds to real/complex/quaternion matrices

At $\theta = 0$ these are monomial symmetric functions

At $\theta = 1$ these are Schur polynomials

James and Muirhead were advocating the study of Jack polynomials (mostly interested in $\theta = \frac{1}{2}$ real case, relevant for statistics) through second order differential operators.

Annals of Mathematical Statistics

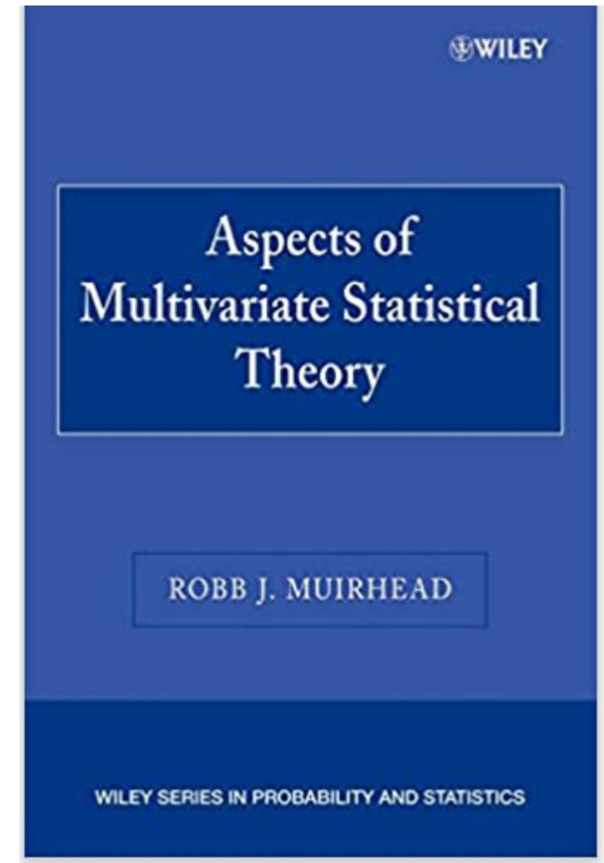
Info | All issues | Search

Ann. Math. Statist.
Volume 39, Number 5 (1968), 1711-1718.

← Previous article | TOC | Next article →

Calculation of Zonal Polynomial Coefficients by Use of the Laplace-Beltrami Operator

A. T. James



Later Sekiguchi found operators of all (rather than second) orders diagonalized by Jack's.

Publ. RIMS. Kyoto Univ.
12 Suppl. (1977), 455–464.

Zonal Spherical Functions on Some Symmetric Spaces

by

Jiro SEKIGUCHI*

Ian G. Macdonald

Mathematician

Ian Grant Macdonald FRS is a British mathematician known for his contributions to symmetric functions, special functions, Lie algebra theory and other aspects of algebra, algebraic combinatorics, and combinatorics. He was educated at Winchester College and Trinity College, Cambridge, graduating in 1952.
Wikipedia

Born: October 11, 1928 (age 92 years), London, United Kingdom

That was the starting point for the breakthrough of Macdonald:

He aimed to define a common generalization of Hall-Littlewood and Jack polynomials by generalizations (discrete versions) of Sekiguchi operators.

Publ. I.R.M.A. Strasbourg, 1988, 372/S–20
Actes 20e Séminaire Lotharingien, p. 131–171

A NEW CLASS OF SYMMETRIC FUNCTIONS

BY

I. G. MACDONALD

Contents.

Central idea for Macdonald operators

Recall from the last lecture $\mathcal{D}_q^N = \sum_{i=1}^N \left[\prod_{j \neq i} \frac{q x_i - x_j}{x_i - x_j} \right] T_{q,i}$

Macdonald: in fact, we can take these two parameters to be different!

Definition: Choose (and fix forever) parameters (q,t)

The first Macdonald operator in N variables is

multiplies the i-th variable by q

$$D_1^N = \sum_{i=1}^N \left[\prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} \right] T_{q,i}$$

More generally, the k-th Macdonald operator is

$$D_k^N = t^{\frac{k(k-1)}{2}} \sum_{I \subset \{1,\dots,N\},\, |I| = k} \left[\prod_{i \in I,\, j \notin I} \frac{t x_i - x_j}{x_i - x_j} \right] \prod_{i \in I} T_{q,i}$$

Definition: Macdonald polynomials $P_\lambda(x_1,\dots,x_N;q,t)$ are simultaneous eigenfunctions of D_k^N:

$$D_k^N P_\lambda = e_k(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \dots, q^{\lambda_N}) \cdot P_\lambda \qquad \lambda \in \mathbb{Y}$$

$[k=1 \rightsquigarrow \sum_{i=1}^{N} q^{\lambda_i} t^{N-i}]$

with normalization

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} (\text{coefficient}) \cdot m_\mu$$

dominance order, $|\mu| = |\lambda|$

Our next task: To make sense of this definition by showing that D_k^N indeed have eigenfunctions (= eigenvectors) of such kind.

Step: It is convenient to work with all $D^N_K,\ 1\le K\le N$ simultaneously by using their generating series

$$D^N(z)=1+\sum_{K=1}^{N} D^N_K\cdot z^K$$

Proposition 1: $$D^N(z)=\prod_{i<j}(x_i-x_j)^{-1}\sum_{\sigma\in S_N}(-1)^\sigma x_1^{N-\sigma(1)}\cdot\ldots\cdot x_N^{N-\sigma(N)}\cdot\prod_{i=1}^{N}\left(1+z\,t^{N-\sigma(i)}T_{q,i}\right) \quad (*)$$

Proof: Let us compute the coefficient of z^K in $(*)$

$K=0$: $$\prod_{i<j}(x_i-x_j)^{-1}\sum_\sigma(-1)^\sigma\prod_i x_i^{N-\sigma(i)}=1.$$

$K=1$: $$\sum_{i=1}^{N}\left[\prod_{a<b}(x_a-x_b)^{-1}\sum_\sigma(-1)^\sigma x_1^{N-\sigma(1)}\cdot\ldots\cdot x_N^{N-\sigma(N)}\cdot t^{N-\sigma(i)}T_{q,i}\right]=$$

apply only to this

$$=\sum_{i=1}^{N}\left[\prod_{a<b}(x_a-x_b)^{-1}\cdot\left(T_{t,i}\prod_{a<b}(x_a-x_b)\right)\right]T_{q,i}=\sum_{i=1}^{N}\prod_{j\neq i}\frac{tx_i-x_j}{x_i-x_j}T_{q,i}=D^N_1$$

$K>1$:

$$\sum_{\substack{I\subset\{1,..N\}\\|I|=K}} \prod_{a<b}(x_a-x_b)^{-1}\cdot\left[\left(\prod_{i\in I}T_{t,i}\right)\cdot\prod_{a<b}(x_a-x_b)\right]\cdot\prod_{i\in I}T_{q,i}$$

apply only to this

$$=\sum_{\substack{I\subset\{1,..N\}\\|I|=K}} t^{\frac{K(K-1)}{2}}\cdot\prod_{i\in I}\prod_{j\notin I}\frac{tx_i-x_j}{x_i-x_j}\prod_{i\in I}T_{q,i}$$

appears from factors like $\frac{tx_i-tx_j}{x_i-x_j}=t$

$$= D_K^N$$

Proposition 2:
$$D^N(z)\,m_\lambda=\sum_{\sigma\in S_N}\prod_{i=1}^{N}\left(1+z\,t^{n-i}\,q^{\lambda_{\sigma(i)}}\right)\frac{s_{\sigma(\lambda)}}{|S_N^\lambda|},$$

where
$$s_{\sigma(\lambda)}=\frac{\det\left[x_i^{\lambda_{\sigma(j)}+N-j}\right]_{i,j=1}^{N}}{\prod_{i<j}(x_i-x_j)}$$

and $|S_N^\lambda|$ = # permutations in S_N fixing λ

[Example: if $\lambda=1^M0^{N-M}$, then $|S_N^\lambda|=M!\,(N-M)!$]

Proof: $D^N(z)\left[x_1^{\nu_1} \cdot x_2^{\nu_2} \cdot \ldots \cdot x_N^{\nu_N}\right] = \prod_{i<j} (x_i - x_j)^{-1}$

$$\sum_{\sigma_1 \in S_N} (-1)^{\sigma_1} \prod_{i=1}^{N} \left(1 + z\, t^{N-\sigma_1(i)} q^{\nu_i}\right) \cdot x_1^{\nu_1 + N - \sigma_1(1)} \cdot \ldots \cdot x_N^{\nu_N + N - \sigma_1(N)}$$

Now take another permutation $\sigma_2 \in S_N$ and sum over all $(\nu_1, \nu_2, \ldots, \nu_N) = (\lambda_{\sigma_2(1)}, \ldots \lambda_{\sigma_2(N)})$

Define the third permutation σ by $\sigma_2 = \sigma_1 \sigma$

We get: $|S_N^\lambda| D^N(z)[m_\lambda] = \prod_{i<j} (x_i - x_j)^{-1} \cdot$

$$\cdot \sum_{\sigma_1, \sigma_2} (-1)^{\sigma_1} \prod_{i=1}^{N} \left(1 + z\, t^{N-i} q^{\lambda_{\sigma(i)}}\right) \cdot x_1^{\lambda_{\sigma_1\sigma(1)} + N - \sigma_1(1)} \cdot \ldots \cdot x_N^{\lambda_{\sigma_1\sigma(N)} + N - \sigma_1(N)}$$

$$= \sum_{\sigma} \prod_{i=1}^{N} \left(1 + z\, t^{N-i} q^{\lambda_{\sigma(i)}}\right) S_{\sigma(\lambda)}(x_1, \ldots x_N)$$

Corollary: $$D^N(z)\, m_\lambda = \prod_{i=1}^{N} \left(1 + z\, q^{\lambda_i} t^{N-i}\right) m_\lambda +$$

$$(**) \qquad + \sum_{\mu < \lambda} (\text{coefficient}) \cdot m_\mu$$

dominance order, $|\mu| = |\lambda|$

Proof: In proposition 2 there are $|S_n^\lambda|$ permutations σ, which fix λ, so that $s_{\sigma(\lambda)} = s_\lambda$. Expanding $s_\lambda = m_\lambda + (\ldots)$, this gives the coefficient of m_λ in $(**)$.

If σ does not fix λ, then $s_{\sigma(\lambda)} = \pm s_\nu$ for some $\nu < \lambda$, since $(\mu_i + N - i)$ is the rearrangement in the increasing order of $(\lambda_{\sigma(i)} + N - i)$. Expanding s_ν in monomial symmetric functions m_μ, we get the desired statement. $\blacksquare$

Theorem: For each $K=1,2,\dots N$, the operator D_K^N has a basis (in Λ_N) of eigenvectors P_λ, such that

$$D_K^N P_\lambda = e_K(q^{\lambda_1}t^{N-1},\dots,q^{\lambda_N})\, P_\lambda \quad \text{and}$$

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} (\text{coefficient}) \cdot m_\mu$$

dominance order, $|\mu|=|\lambda|$.

Remark: So far we do not claim that P_λ do not depend on K — this will be shown in the next lecture.

Proof: Taking the coefficient of z^κ in both sides of the last corollary, we get $D_K^N m_\lambda = e_K(q^{\lambda_1}t^{N-1},\dots,q^{\lambda_N})\, m_\lambda + \sum_{\mu<\lambda} (\dots)\, m_\mu$. Hence, D_K^N is triangular in m_λ basis with $e_K(q^{\lambda_i}t^{N-i})$ on the diagonal.

Therefore, eigenvalues of D_K^N are $e_K(q^{\lambda_i}t^{N-i})$ and P_λ are the eigenvectors. ∎