Three main topics:

1) Algebras Λ_N and Λ, their generators

2) Schur functions s_λ: equivalent defs, branching rules, summation formulas

3) Combinatorics of Young diagrams: Row/length duality, dimensions, RSK

Λ_N - symmetric functions in $(x_1, .. x_N)$

Λ - symmetric power series in ∞-many vars $(x_1, x_2, ..)$ of bounded degree

$\rho_{N,N-1} : \Lambda_N \to \Lambda_{N-1} : x_N = 0$

$\rho_N : \Lambda \to \Lambda_N : x_{N+1} = x_{N+2} = .. = 0$

$\Lambda_1 \leftarrow \Lambda_2 \leftarrow \;..\; \leftarrow \Lambda$

projective limit of graded algebras

Any identity in Λ = limit of identities in Λ_N.

Key theorem:

$$\Lambda = \mathbb{R}[e_1, e_2, \dots] = \mathbb{R}[h_1, h_2, \dots]$$
$$= \mathbb{R}[p_1, p_2, \dots] = \langle m_\lambda \rangle_{\lambda \in Y}$$

$$e_k = \sum_{i_1 < \dots < i_k} x_{i_1} \cdot \ldots \cdot x_{i_k} \qquad h_k = \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \cdot \ldots \cdot x_{i_k}$$

$$p_k = \sum_i (x_i)^k \qquad m_\lambda = \mathrm{Sym}\left(x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n}\right)$$

Key skill: expand symmetric polynomials in m_λ, or e_λ, or h_λ, or p_λ

In particular, we know how to re-expand

$$E(t) = \sum_{k \geq 0} e_k t^k \quad \Big| \quad H(t) = \sum_{k \geq 0} h_k t^k \quad \Big| \quad P(t) = \sum_{k \geq 1} \frac{p_k t^k}{k}$$

$H(t)\, E(-t) = 1$ $\qquad$ $H(t) = \exp(P(t))$

Polynomial relations between e_k, h_k, p_k

Important automorphism: $\omega : \Lambda \to \Lambda$

$\omega(e_k) = h_k$ $\qquad$ $\omega(p_k) = (-1)^{k-1} p_k$

$\omega(h_k) = e_k$ $\qquad$ $\omega^2 = \mathrm{Id}$

Questions?

Schur functions s_λ.

$\lambda \in \mathbb{Y}$ $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$

Five equivalent definitions

A) $$s_\lambda(x_1, \dots x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^{N}}{\prod\limits_{i<j}(x_i - x_j)}$$

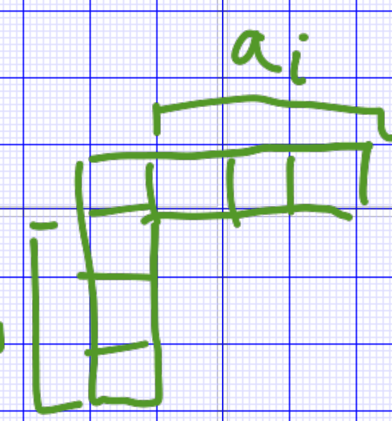
B) $$s_\lambda = \det\left[h_{\lambda_i - i + j}\right]_{i,j=1}^{n}, \quad n \geq \ell(\lambda)$$

$h_0 = 1$

$h_k = 0,\ k<0$

$$= \det\left[e_{\lambda'_i - i + j}\right]_{i,j=1}^{n}, \quad n \geq \lambda_1$$

C) $$s_\lambda = \det\left[s_{(a_i|b_j)}\right]_{i,j=1}^{r}$$



$a_i = \lambda_i - i$, $b_j = \lambda'_j - j$, r – lenght of diagonal of λ

D) $$s_\lambda(x_1,\dots x_N) = \sum_{0 \prec \underset{\substack{\cap \\ GT_1}}{\lambda^1} \prec \dots \prec \underset{\substack{\cap \\ GT_N}}{\lambda^N} = \lambda} \prod_{i=1}^{N} (x_i)^{|\lambda^i| - |\lambda^{i-1}|}$$

$\mu \prec \lambda$ if $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots$

GT – Gelfand – Tsetlin

$$GT_N = \{\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \mid \lambda_i \in \mathbb{Z}\}$$

SSYT

$$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 3 & 3 & 3 & 3 \\ 2 & 3 & 5 \\ 3 & 4 \\ 4 & 6 \\ 5 \\ 6 \end{array}$$

Part filled with $\leq k$ is λ^k

$\lambda^2 = (4,1)$

$|\lambda^k| - |\lambda^{k-1}| = \#$ of k in SSYT

E) $s_\lambda = m_\lambda$ + (lower degree terms)

in lexicographic / dominance order

$$\left[x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N} \right]$$

and $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda=\mu}$

$$\langle f, g \rangle_N = \frac{1}{(2\pi i)^N N!} \int_{|z_i|=1} \cdots \int \prod_{i<j} |z_i - z_j|^2 f(z_1, \ldots z_N) \cdot g(z_1^{-1}, \ldots z_N^{-1}) \cdot \frac{dz_1}{z_1} \cdots \frac{dz_N}{z_N}$$

$$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda=\mu}$$

$\lambda_1 \geq \lambda_2 \geq \ldots$

m_i = # parts i in λ

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda=\mu} \cdot z_\lambda \qquad z_\lambda = \prod_{i \geq 1} i^{m_i} \cdot m_i!$$

Key skill: expand a polynomial in s_λ

The most useful approach:

$$f(x_1, \dots x_N) = \sum_\lambda c_\lambda s_\lambda(x_1, \dots x_N)$$

$$c_\lambda = [x_1^{\lambda_1+N-1} \cdot \dots \cdot x_N^{\lambda_N}] \left(f \cdot \prod_{i<j} (x_i - x_j) \right)$$

coefficient of monomial

Key identities: $\omega(s_\lambda) = s_{\lambda'}$

$$\sum_{\lambda \in \mathbb{Y}} s_\lambda(x_1, \dots) s_\lambda(y_1, \dots) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j} =$$

$$= \prod_i H_{\vec{x}}(y_i) = \prod_i H_{\vec{y}}(x_i) = \exp\left(\sum_{k \geq 1} \frac{p_k(x_1, \dots) \, p_k(y_1, \dots)}{k} \right)$$

Key applications: Representations of S_n, $U(N)$

Counting of plane partitions / 3d Young diagrams / non-intersecting paths / lozenge tilings

Important tool: specializations

$$\rho: \Lambda \to \mathbb{R}$$

Schur-positive specializations

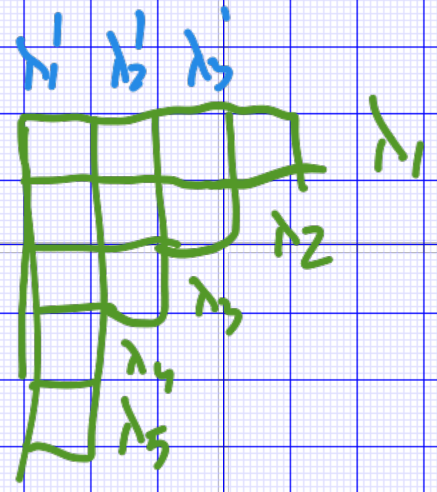
$$\left(\alpha_1 \geq \alpha_2 \geq \ldots \geq 0 \; ; \; \beta_1 \geq \beta_2 \geq \ldots \geq 0 \; ; \; \gamma \geq 0\right)$$

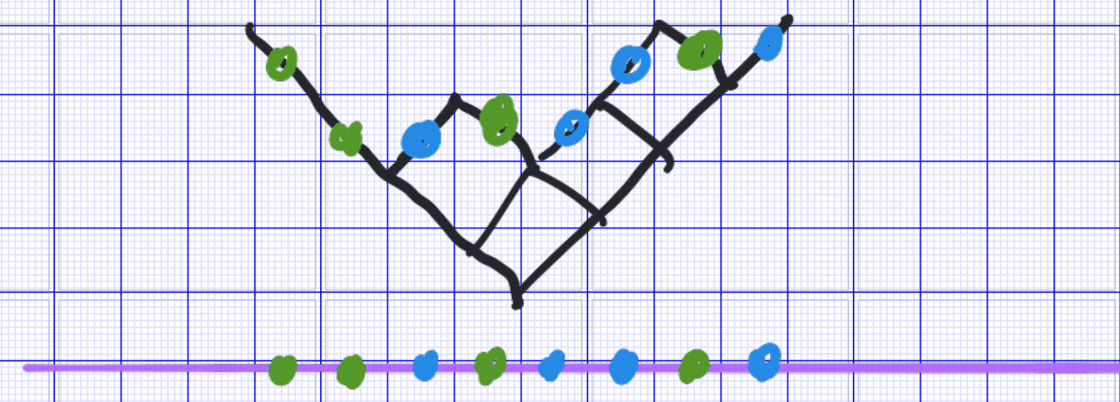
$$\sum(\alpha_i + \beta_i) < \infty$$

Important generalization: skew Schur functions

$$s_{\lambda/\mu}$$

Combinatorics of Young diagrams





$\#$SSYT of given shape $=$ $\#$GT-patterns $=$

$=$ $\#$lozenge tilings of trapezoid $= \dim_N(\lambda) =$

$$= s_\lambda(1^N) = \prod_{i<j} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}$$

$$\# \text{SYT} = \dim\lambda = \langle s_\lambda, p_1^n \rangle =$$

$$= \frac{n!}{\prod_{(i,j)\in\lambda} h(i,j)} = \frac{n!}{\prod_{i=1}^{N}(\lambda_i+N-i)!} \prod_{i<j\leq N}\left((\lambda_i-i)-(\lambda_j-j)\right)$$

$$n! = \sum_\lambda (\dim\lambda)^2$$

rep-theoretic meaning

bijective proof

$$\pi \xleftrightarrow{\text{Robinson-Schensted}} (P, Q)$$

$\pi \in S_n$

(P, Q) SYT of same shape $\lambda \in \mathbb{Y}_n$

λ_1 = Longest Increasing Subsequence in π

Generalization : RSK

(integer matrices) $\longleftrightarrow$ pairs of SSYT of same shape