Math 740 Specializations Lecture 9

Formal identities in Λ

Specializations $\downarrow$ $\uparrow$ Taylor expansion; number of variables $N \to \infty$

Numeric / analytic identities
(with numbers, real sums, integrals, etc)

Example: We know $\sum_{\lambda \in \mathbb{Y}} s_\lambda(x_1, \dots) s_\lambda(y_1, \dots) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}$ (*)

We want to prove the Burnside identity:

$$\sum_{\lambda \in \mathbb{Y}_n} (\dim \lambda)^2 = n! \qquad (**)$$

[Sum of squares of dimensions of ir. reps = size of the group = $|S_n|$]

Claim: (**) follows from (*) by a specialization.

Definition: A specialization is an algebra homomorphism

$$\rho : \Lambda \to \mathbb{R}$$

$\rho(\lambda) = \lambda \quad , \lambda \in \mathbb{R} \subset \Lambda$

$\rho(a+b) = \rho(a) + \rho(b)$

$\rho(ab) = \rho(a)\rho(b)$

Definition: A specialization ρ is called Schur-positive if $\forall \lambda \in \mathbb{Y}$ $\quad \rho(S_\lambda) \geq 0$

Example 1: Take a sequence of reals $\alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq \dots \geq 0$ with $\sum_i \alpha_i < \infty$ and plug in $x_i \to \alpha_i$, which means

$$\rho_\alpha(p_k) = \sum_{i \geq 1} (\alpha_i)^k$$

[Since $\Lambda = \mathbb{R}[p_1, p_2, \dots]$, $\rho(p_k)$ uniquely determine ρ]

Since $S_\lambda = \sum_\mu \underset{\uparrow\ \mathbb{Z}_{\geq 0}}{K_\lambda^\mu}\, m_\mu$ and $\rho(m_\mu) \geq 0$, this specialization is Schur-positive

Example 2: Take a sequence of reals $\beta_1 \geq \beta_2 \geq \ldots \geq 0$ with $\sum_i \beta_i < \infty$ and set $\rho_\beta(p_k) = (-1)^{k-1} \sum_{i=1}^{\infty} (\beta_i)^k$

This is Schur positive, because ρ_β is a composition of Example 1 and involution w:

$$\rho_\beta(s_\lambda) = (w(s_\lambda))(\beta_1, \beta_2, \ldots) = s_{\lambda'}(\beta_1, \beta_2, \ldots)$$

substitution as in Example 1

Example 3: Take $\gamma \geq 0$, and define

$$\rho_\gamma(p_k) = \begin{cases} \gamma, & k = 1, \\ 0, & k > 1. \end{cases}$$

This is Schur-positive, because it can be represented as $N \to \infty$ limit of $p_k \to \sum_{i=1}^{N} \left(\frac{\gamma}{N}\right)^k$, which fits into Example 1.

Theorem: All Schur-positive specializations are parameterized by $\alpha_1 \geq \alpha_2 \geq \ldots \geq 0$, $\beta_1 \geq \beta_2 \geq \ldots \geq 0$, $\gamma \geq 0$ with $\sum_i \alpha_i + \sum_i \beta_i < \infty$ and are given by

$$\rho_{(\alpha,\beta,\gamma)}(p_k) = \begin{cases} \sum_i \alpha_i + \sum_i \beta_i + \gamma, & k=1, \\ \sum_i (\alpha_i)^k + (-1)^{k-1} \sum_i (\beta_i)^k, & k>1. \end{cases}$$

We do not provide a proof. There are several equivalent formulations

$\rho_{(\alpha,\beta,\gamma)}(s_\lambda) \geq 0$ – not hard

no other Schur positive – challenging

- Classification of extreme characters of finite factor representations of infinite symmetric group S_∞

Published: February 1964

Die unzerlegbaren, positiv-definiten Klassenfunktionen der abzählbar unendlichen, symmetrischen Gruppe

Elmar Thoma

Mathematische Zeitschrift **85**, 40–61(1964) | Cite this article

Proc Natl Acad Sci U S A. 1951 May; 37(5): 303–307. PMCID:

doi: 10.1073/pnas.37.5.303 PM

On the Generating Functions of Totally Positive Sequences

Michael Aissen, Albert Edrei, I. J. Schoenberg, and Anne Whitney

Canadian Journal of Mathematics Search

Article Metrics

Volume 5 1953 , pp. 86-94 Cited by 27 Access

Proof of a Conjecture of Schoenberg on the Generating Function of a Totally Positive Sequence

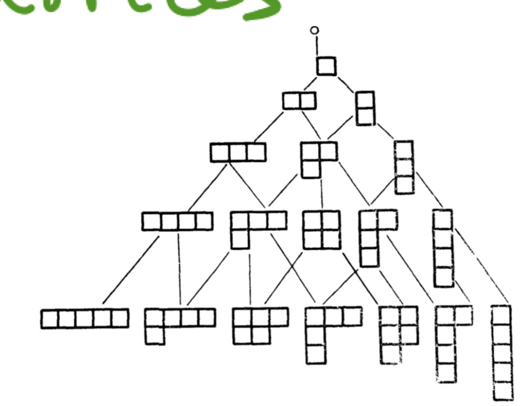
Albert Edrei (a1)

- Classification of totally positive triangular Toeplitz matrices

Asymptotic theory of characters of the symmetric group

A. M. Vershik & S. V. Kerov

Functional Analysis and Its Applications **15**, 246–255(1981) | Cite this article



- Identification of the boundary of the Young graph

$\rho_\gamma\ [p_1 \to \gamma;\ p_k \to 0,\ k>1]$ is called Planchérel specialization

Proposition: $\rho_\gamma(s_\lambda) = \dfrac{\gamma^{|\lambda|}}{|\lambda|!} \dim\lambda$

Proof Recall from Lecture 7 that

$\dim\lambda = \chi^\lambda_{1^{|\lambda|}}$, where χ^λ_ρ is found from expansion

$s_\lambda = \sum_\rho \chi^\lambda_\rho \dfrac{p_\rho}{z_\rho}$ ← Apply ρ_γ to get only one non-zero term $(\rho = 1^{|\lambda|})$

$\rho_\gamma(s_\lambda) = \dfrac{\overset{\chi^\lambda_{1^{|\lambda|}}}{\dim\lambda}}{\underset{z_{1^{|\lambda|}}}{|\lambda|!}} \underset{\rho_\gamma(p_1^{|\lambda|})}{\gamma^{|\lambda|}}$ $\blacksquare$

Corollary: $\sum_{\lambda \in \mathbb{Y}_n} (\dim\lambda)^2 = n!$

Proof of Corollary

[see 2nd slide of Lec. 8 or take log and Taylor expand]

$$\sum_\lambda s_\lambda(x_1,\dots)\, s_\lambda(y_1,\dots) = \prod_{i,j} \frac{1}{1-x_i y_j} = \exp\left(\sum_{k\geq 1} \frac{p_k(x_1,\dots)\, p_k(y_1,\dots)}{k}\right)$$

Apply ρ_γ in x-variables and ρ_1 in y-variables

$$\sum_\lambda \left(\frac{\gamma^{|\lambda|}}{|\lambda|!}\dim\lambda\right)\cdot\left(\frac{\dim\lambda}{|\lambda|!}\right) = \exp(\gamma) = \sum_{n\geq 0}\frac{\gamma^n}{n!}$$

Compare the coefficients of γ^n in both sides to get

$$\sum_{\lambda\,:\,|\lambda|=n} \frac{(\dim\lambda)^2}{(n!)^2} = \frac{1}{n!} \qquad \blacksquare$$

Remark: In representation theory of S_n, the identity is a shadow of

$$\bigoplus_{\lambda\in\mathbb{Y}_n} V_\lambda \otimes \widehat{V}_\lambda = L_2(S_n)$$

decomposition of biregular representation ($S_n\times S_n$ acting on functions on S_n)

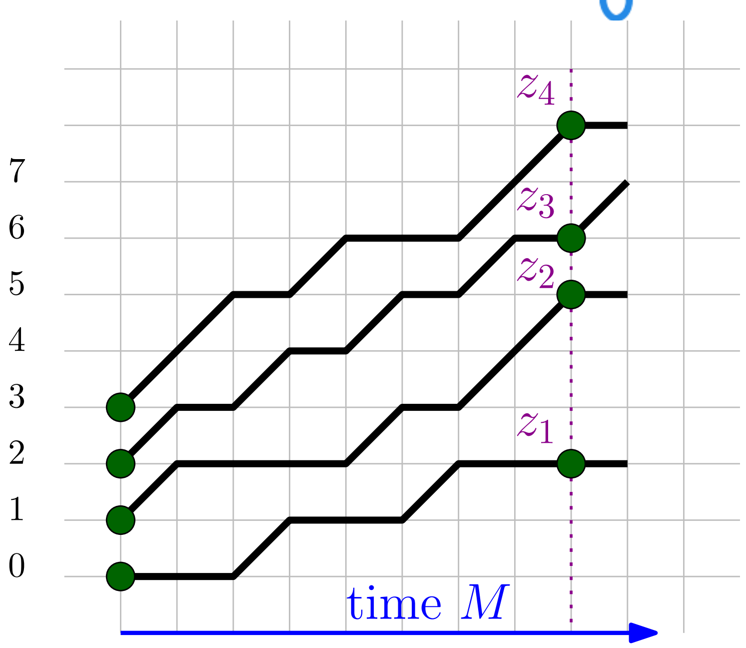
Here is another corollary.

Theorem:
$$\sum_{0 \le z_1 < z_2 < \dots < z_N \le N+M-1} \alpha^{\sum_i z_i} \cdot \prod_{i<j} (z_i - z_j)^2 \cdot \prod_{i=1}^{N} \binom{N+M-1}{z_i}$$
$$= (1+\alpha)^{MN} \cdot \prod_{i=0}^{N-1} \frac{i!}{(M+i)!} \cdot \left[(N+M-1)!\right]^N \cdot \alpha^{\frac{N(N-1)}{2}}$$

- $N=1$: $\displaystyle\sum_{0 \le z \le M} \alpha^z \binom{M}{z} = (1+\alpha)^M$ Binomial theorem

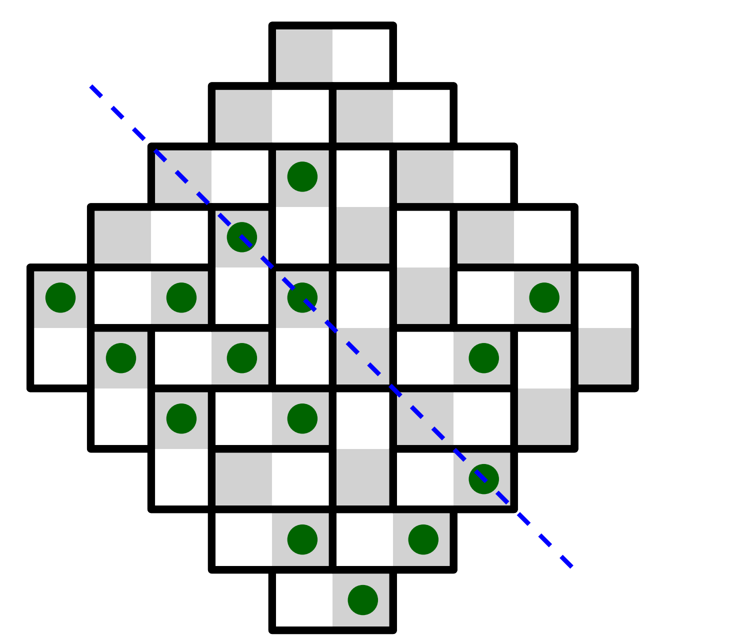
[another multidimensional binomial theorem in HW2, Problem 5]

- In probability, this is the distribution of random $\{z_i\}_{i=1}^N$ in non-intersecting random walks or sections of tilings of the Aztec diamond



Called Krawtchouk ortogonal polynomial ensemble

because of the relevance of Krawtchouk polynomials for the asymptotic analysis.



Proof of theorem.

generating function of $h_k(y_1, y_2, \dots)$

$$\sum S_\lambda(x_1,\dots)\, S_\lambda(y_1,\dots) = \prod \frac{1}{1-x_i y_j} = \prod_i H_{y_1,y_2\dots}(x_i)$$

Apply ρ with $\alpha_1=\alpha_2=\dots=\alpha_N=\alpha$ in x-variables

Apply $\tilde{\rho}$ with $\beta_1=\beta_2=\dots=\beta_M=1$ in y-variables

$$\text{RHS} \to \prod_{i=1}^{N} \left(1 + \tilde{\rho}(h_1)\alpha + \tilde{\rho}(h_2)\alpha^2 + \dots\right)$$

Recall that $\tilde{\rho}$ is composition of ω and substitution of M variables equal to 1. Since $\omega(H(t)) = E(t) = \prod(1+ty_i)$ we get

$$\text{RHS} \to (1+\alpha)^{MN}$$

LHS: $\rho(S_\lambda(x_1,\dots)) = \alpha^{|\lambda|} S_\lambda(1^N) = \alpha^{|\lambda|} \prod_{1\le i<j\le N} \frac{(\lambda_i - i) - (\lambda_j - j)}{j-i}$

$$\tilde{\rho}(S_\lambda(y_1,\dots)) = S_{\lambda'}(1^M) = \prod_{1\le i<j\le M} \frac{(\lambda'_i - i) - (\lambda'_j - j)}{j-i}$$

$$\boxed{\begin{array}{l} \lambda_{N+1}=0 \\ \lambda'_{M+1}=0 \end{array} \Rightarrow \begin{array}{l} \lambda_1 \le M \\ \lambda'_1 \le N \end{array}}$$

We now use the duality between rows / columns via particle-hole duality as in Lecture 7 (slide 3)

Set $z_i = \lambda_{N+1-i} + (i-1)$ $\quad [0 \le z_1 < z_2 < \dots \le M+N-1]$

$\{u_i\} = \{0,1,\dots,M+N-1\} \setminus \{z_i\}$

$$\mathrm{LHS} = q^{\sum_{i=1}^{N} z_i} \prod_{1\le i<j\le N} (z_j - z_i) \prod_{1\le i<j\le M} (u_j - u_i) \frac{q^{\sum_{i=1}^{N} -(i-1)}}{1!\,2!\dots(N-1)!\;\;1!\,2!\dots(M-1)!}$$

Lemma: $$\prod_{1\le i<j\le M} (u_j - u_i) = \prod_{1\le i<j\le N} (z_j - z_i) \prod_{i=1}^{N} \frac{1}{z_i!\,(M+N-1-z_i)!} \cdot \prod_{0\le a<b\le M+N-1} (b-a)$$

Proof: Write $z_i!\,(M+N-1-z_i)!$ as $\prod_{0\le c\le M+N-1,\; c\ne z_i} |c - z_i|$ and match factors ∎

Hence LHS =

$$\alpha^{\sum_{i=1}^{N} z_i} \cdot \alpha^{-\frac{N(N-1)}{2}} \cdot \prod_{i<j} (z_i - z_j)^2 \cdot \prod_{i=1}^{N} \binom{M+N-1}{z_i} \cdot \left[(M+N-1)!\right]^{-N} \frac{1!\,2! \ldots (M+N-1)!}{1!\,2! \ldots (N-1)!\; 1!\,2! \ldots (M-1)!}$$

Since also RHS $= (1+\alpha)^{MN}$, LHS = RHS gives the statement of the theorem $\blacksquare$

Corollary: (Mehta-Dyson-Selberg integral)

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i<j} (t_i - t_j)^2 \prod_{i=1}^{N} \exp\left(-\frac{t_i^2}{2}\right) dt_1 \cdot \ldots \cdot dt_N$$

$$= (2\pi)^{N/2} \cdot 1! \cdot 2! \cdot \ldots \cdot N!$$

Proof: Set $\alpha = 1$ $\quad z_i = \frac{M}{2} + \frac{\sqrt{M}}{2} t_i$ in theorem, $M \to \infty$

$$\binom{M+N-1}{z_i} \approx 2^{M+N-1} \sqrt{\frac{2}{\pi M}} \exp\left(-\frac{t_i^2}{2}\right)$$

by Stirling's formula or by Central Limit Theorem for sums of i.i.d. Bernoulli

Exercise: fill in the remaining details. $\blacksquare$

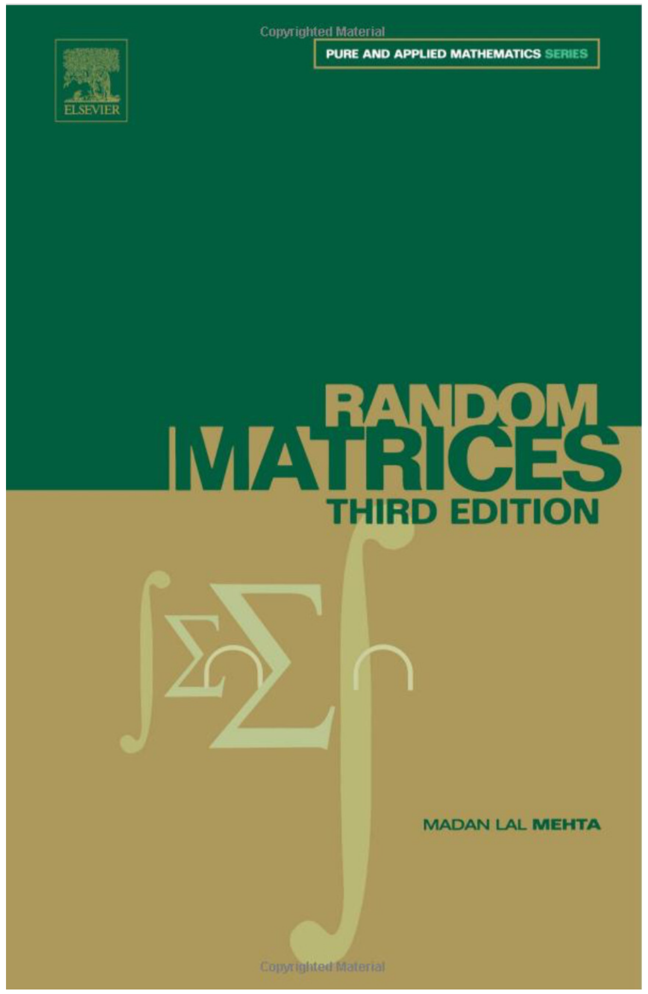
Background for the last integral:

$$\frac{1}{Z} \prod_{i<j} (t_i - t_j)^2 \prod_{i=1}^{N} \exp\left(-\frac{t_i^2}{2}\right) \qquad (*)$$

normalizing constant to make total mass equal to 1

is the density of

Gaussian Unitary Ensemble



Theorem: Take X – $N \times N$ complex matrix whose matrix elements are independent $N(0,1) + i\, N(0,1)$

Gaussian

mean

variance

Then the distribution of eigenvalues of Hermitian matrix $\frac{1}{2}(X + X^*)$ has density $(*)$.

transpose + conjugate

Proof is given in almost every book on random matrices.