Math 740 Robinson-Schensted-Knuth correspondence Lecture 12

Recall the identity $\sum_{\lambda \in \mathbb{Y}_n} (\dim \lambda)^2 = n!$

Since $\dim \lambda$ = # standard Young tableaux of shape λ:

$\left\{ \begin{array}{c} \text{pairs of standard Young tableaux of} \\ \text{same shapes with } n \text{ boxes} \end{array} \right\}$

$\cong$ (same number of elements)

permutations of $1, 2, \ldots, n$

Robinson-Schensted algorithm gives an explicit bijection

Basic ingredient of RS algorithm is row insertion.

Take a semistandard Young tableau P
and a number $x \in \{1, 2, 3, \dots\}$.

We will construct a new tableau P'.

For that set $k := 1$ and repeat the following:

- Look at the k-th row. If x is greater [or equal] than all elements in this row, then insert x at the end of the row and stop.

  Otherwise, let y be the smallest element in row k which is greater than x [choose the leftmost if there are several]
  Replace y with x, set $k := k+1$, $x := y$, repeat •.
  „bump y"

Robinson - Schensted algorithm:

Take a permutation in one-row notation (like 51324)

Start from the empty Young diagram / tableau and sequentially (from left to right) insert all numbers.

Final output P — insertion tableau

second tableau Q — recording tableau:

records the order in which the new boxes were added to the Young diagram (of insertion tableau) in our process

Result: Map from permutations of 1, 2, ..., n to pairs of Young tableaux of same shape.

Example 1: 1 4 3 6 2 5

1) $\begin{array}{|c|}\hline 1\\\hline\end{array}$

2) Insert 4 into the 1st row $\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}$

3) Insert 3 into the 1st row bumping 4 $\begin{array}{|c|c|}\hline 1&3\\\hline\end{array} \leftarrow 4$

   Insert 4 into the 2nd row $\begin{array}{cc}1&3\\4&\end{array}$

4) Insert 6 into the 1st row $\begin{array}{ccc}1&3&6\\4&&\end{array}$

5) Insert 2 into the 1st row, bumping 3 $\begin{array}{ccc}1&2&6\\4&&\end{array} \leftarrow 3$

   Insert 3 into the 2nd row, bumping 4 $\begin{array}{ccc}1&2&6\\3&&\end{array} \leftarrow 4$

   Insert 4 into the 3rd row $\begin{array}{ccc}1&2&6\\3&&\\4&&\end{array}$

6) Insert 5 into the 1st row, bumping 6 $\begin{array}{ccc}1&2&5\\3&&\\4&&\end{array} \leftarrow 6$

   Insert 6 into the 2nd row $\begin{array}{ccc}1&2&5\\3&6&\\4&&\end{array}$ ← This is the final P

Example 1 (continued) In our process the Young diagram grew as: 

The recording tableau Q records this growth.

1 is here because this box was added at the 1st step

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array}$$

6 is here because this box was added at the 6th step.

Conclusion: $143625 \xrightarrow{RS} \left(\begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 6 & \\ \cline{1-2} 4 & & \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 6 & \\ \cline{1-2} 5 & & \\ \cline{1-1} \end{array} \right)$

Example 2: $4236517 \xrightarrow{RS} \left(\begin{array}{|c|c|c|c|}\hline 1 & 3 & 5 & 7 \\ \hline 2 & 6 & & \\ \cline{1-2} 4 & & & \\ \cline{1-1} \end{array} \, , \, \begin{array}{|c|c|c|c|}\hline 1 & 3 & 4 & 7 \\ \hline 2 & 5 & & \\ \cline{1-2} 6 & & & \\ \cline{1-1} \end{array} \right)$

Exercise: Check this!

Theorem: $\pi \xrightarrow{RS} (P, Q)$ is a bijection between permutations in S_n and pairs of standard Young tableau of the same shape $\lambda \in Y_n$.
not fixed

JOURNAL ARTICLE

On the Representations of the Symmetric Group

G. de B. Robinson

American Journal of Mathematics
Vol. 60, No. 3 (Jul., 1938), pp. 745-760 (16 pages)
Published By: The Johns Hopkins University Press

Canadian Journal of Mathematics

Search Canadian Journal o

Article

Volume 13 1961 , pp. 179-191

Cited by 340
Access

Longest Increasing and Decreasing Subsequences

C. Schensted (a1)

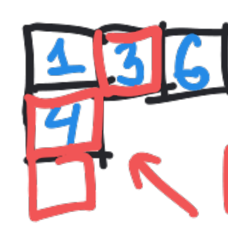
Proof: First, we need to check that P and Q are standard. For Q this is by construction. For P we note that eventually this is a Young diagram with n boxes and filled with numbers $1, 2, .. n$. Hence, it remains to show that the numbers increase in rows/columns on each step.

Rows: we were always inserting numbers, preserving ordering.

Columns: when we insert and bump something, the entry is decreasing $\Rightarrow$ inequality with below entry is preserved. As for the inequality with the above entry: if we are in the 1st row, it does not exist, if we are in ≥ 2nd row, then the inserted element was bumped from the previous row.

Otherwise, it is helpful to notice that the bumping route (positions of entries which are bumped from row to row) is monotone in left-down direction

Recall Step 5 of Example 1:



← insert 2

bumping route goes in left-down direction.

Exercise 1: verify monotonicity of the bumping route

Exercise 2: check that monotonicity implies preservation of inequalities.

Conclusion: we showed that $RS(\pi)$ is a pair (P,Q) of standard Young tableaux.

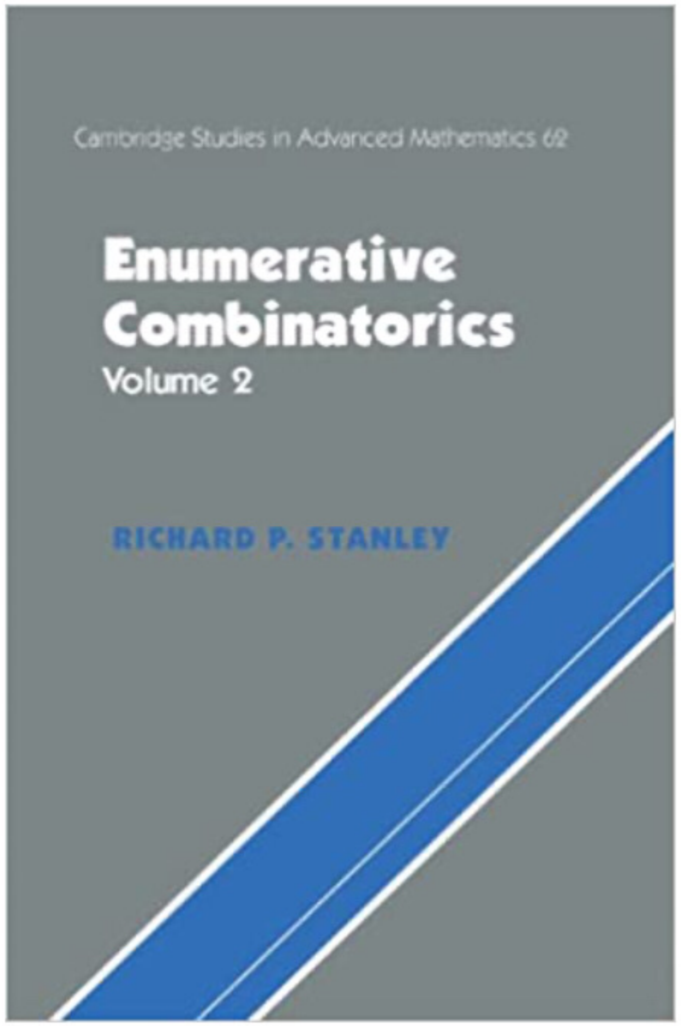
Remains to construct inverse map $RS^{-1}(P,Q) = \pi$

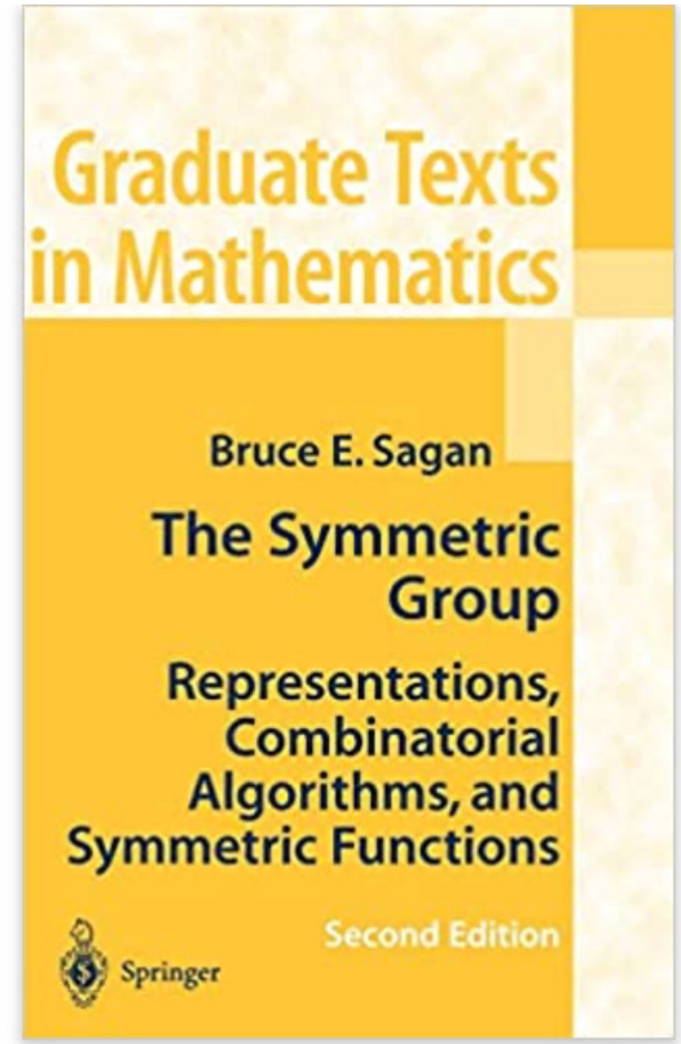
This is done by inverting each step of the algorithm:

- start from (P,Q)
- (*) position of the largest entry n in Q is where the last insertion happened.
- reverse the bumping procedure
  - A) take x in row k (entry at last insertion) Delete it in P
  - B) find the largest element y in row $k-1$ smaller than x
  - C) replace y by x in P. Set $k := k-1$, $x := y$, return to step A)
- When we reach row 1, we get the last element of π, new smaller P and new Q (smaller by removing n). Return to (*)

Important: think through the RS and RS^{-1}.

For more details see:





Robinson-Schested correspondence has very rich structure. We cover only a portion. Here is an example of a nice theorem that we do not prove:

Theorem: If $RS(\pi) = (P, Q)$, then $RS(\pi^{-1}) = (Q, P)$.

RS correspondence has many generalizations.
The first one is due to Schensted.

Theorem: Take N letters $\{1, \dots, N\}$ and consider words of length n in them [there are N^n of them]
Applying the same RS algorithm to words, we get a bijection

same shape λ : $|\lambda| = n$, $\ell(\lambda) \leq N$

$$\text{words of length } n \text{ in letters } 1, \dots, N \xrightarrow{RS} (P, Q)$$

semistandard Young tableau

standard Young tableau

Proof The same argument works: one checks that P remains semistandard on each step of RS-algorithm and that the procedure is invertible. ∎

Corollary: Set $\dim(\lambda) = \#$ Standard Young tableaux of shape λ
(= dimension of S_n-representation labeled by λ)
$\mathrm{Dim}_N(\lambda) = \#$ Semistandard Young tableaux of shape λ in letters $1,..N$
(= dimension of $U(N)$-representation labeled by λ)
[We had explicit formulas for both dimensions previously!]

Then $$\sum_{\lambda \in Y_n \cap GT_N} \dim(\lambda)\,\mathrm{Dim}_N(\lambda) = N^n$$

Proof: RS-correspondence matches sets in RHS and LHS ∎

This is a shadow of a representation theoretic fact.

Theorem (Schur-Weyl duality; we do not give a proof)

S_n permutes factors

$$\left(\mathbb{C}^N\right)^{\otimes n} \cong \bigoplus_{\lambda \in Y_n \cap GT_N} V_\lambda \otimes \widetilde{V}_\lambda$$

representation of $U(N)$

$U(N)$ acts naturally in $\mathbb{C}^N$

representation of S_n

Robinson-Schensted-Knuth correspondence (RSK)

Pacific Journal of Mathematics

Info All issues Search

Pacific J. Math.
Volume 34, Number 3 (1970), 709-727.

← Previous

Permutations, matrices, and generalized Young tableaux.

Donald E. Knuth

A generalized permutation

$$\begin{matrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & & j_n \end{matrix}$$

With $(i_1, j_1) \le (i_2, j_2) \le \ldots \le (i_n, j_n)$

[Lexicographic order, meaning that $i_1 \le i_2 \le \ldots \le i_n$ and $j_k \le j_{k+1}$ whenever $i_k = i_{k+1}$]

Theorem: Let M be an infinite $\mathbb{N} \times \mathbb{N}$ matrix filled with non-negative integers of finite total sum n.

$\pi(M)$ = generalized permutation: $M_{i,j}$ = # of occurences of $\binom{i}{j}$ in $\pi(M)$

$\pi(M) = \begin{pmatrix} i_1 & i_2 & \cdots & i_n \\ j_1 & j_2 & \cdots & j_n \end{pmatrix}$

semistandard Young tableaux of same shapes

Set $RSK(M) = (P, Q)$

from $RS(j_1, \ldots j_n)$

from $RS(j_1, \ldots j_n)$, replacing $k \to i_k$ in Q

Then this is a bijection between matrices and pairs of SSYT.

We do not give a proof, but it is constructed along the same lines as bijectivity of RS algorithm.

Observation: RSK is a bijective face of Cauchy-Littlewood identity

$$\sum_{\lambda} s_\lambda(x_1, x_2, \ldots)\, s_\lambda(y_1, y_2, \ldots) = \prod_{i,j} \frac{1}{1 - x_i y_j} = \prod_{i,j \geq 1} \sum_{k=0}^{\infty} x_i^k y_j^k$$

each one is a sum of monomials, with summation over SSYT of shape λ

(combinatorial formula)

expand product and encode each term as matrix M with $M_{i,j} = k \rightsquigarrow$ factor $x_i^k y_j^k$