Math 740 Lecture 7

Last time: $s_\lambda = \det\left[h_{\lambda_i - i + j} \right]_{i,j=1}^{n}$, $n \geq \ell(\lambda)$.

Theorem (Nägelsbach-Kostka formula):

$$s_\lambda = \det\left[e_{\lambda'_i - i + j} \right]_{i,j=1}^{n}, \quad n \geq \lambda_1.$$

Proof: Fix large enough n and consider the matrices $H = \left[h_{j-i} \right]_{i,j=1}^{n}$ and $E = \left[(-1)^{j-i} e_{j-i} \right]_{i,j=1}^{n}$

They are both uniuppertriangular and $EH = HE = Id$

Indeed, $[HE]_{ik} = \sum_j h_{j-i}\, e_{k-j}\, (-1)^{k-j}$

- If $i = k$, then there is a unique $\neq 0$ term $e_0 \cdot h_0$
- If $i > k$, then all terms $= 0$
- If $i < k$, then the sum $= 0$ by relation between e and h.

Lemma (see linear algebra textbooks):

If A, B are two $n \times n$ matrices with $A = B^{-1}$, then

minor of $A = \pm \frac{1}{\det A} \cdot$ complementary minor of B^t

$I, J \subset \{1, \ldots n\}$ $\quad |I| = |J|$

$$\det [A_{ij}]_{i \in I, j \in J} = \frac{(-1)^{\sum_{i \in I} i - \sum_{j \in J} j}}{\det [A]} \cdot \det [B_{j,i}]_{i \in \bar{I}, j \in \bar{J}}$$

$$\left(\bar{I} = \{1, \ldots n\} \setminus I, \quad \bar{J} = \{1, \ldots n\} \setminus J \right)$$

Now take $n = n_1 + n_2$, $\quad n_1 \geq \ell(\lambda)$, $\quad n_2 \geq \lambda_1$.

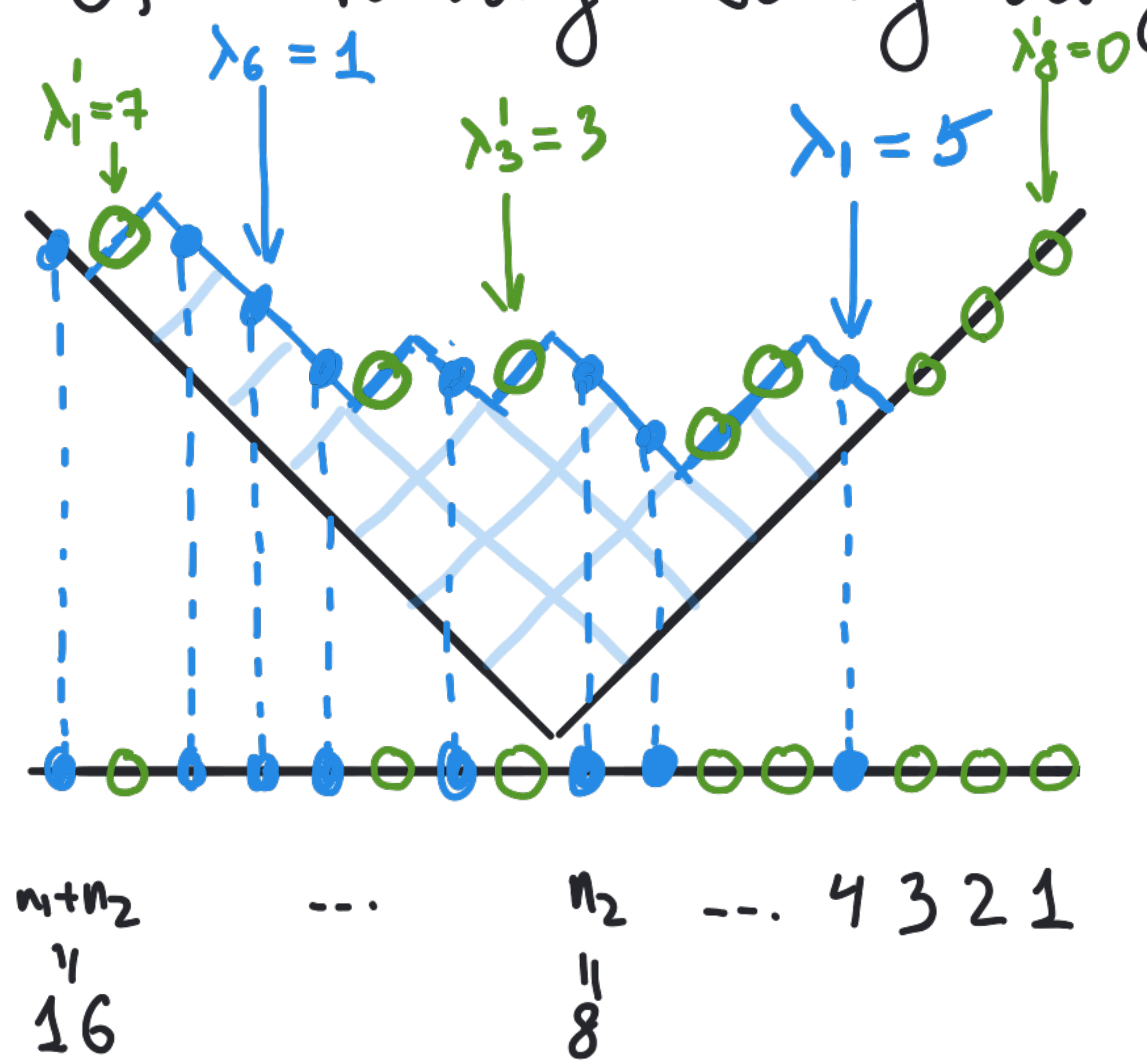
$$I = \{n_2 - \lambda_i + i\}_{i=1}^{n_1} \qquad J = \{n_2 + j\}_{j=1}^{n_1}$$

We need to figure out the complementary minor

(That will finish the proof of the theorem)

Russian style of drawing Young diagrams is helpful

blue particles $\{ n_2 - \lambda_i + i \}$



green holes $\{ \lambda'_i - i + n_2 + 1 \}$

coordinates:

Complement of particles = holes

$$\bar{I} = \{ \lambda'_i - i + n_2 + 1 \}_{i=1}^{n_2} \qquad \bar{J} = \{ n_2 + 1 - j \}_{j=1}^{n_2}$$

Conclusion: $$S_\lambda = \det \left[h_{(n_2+j)-(n_2-\lambda_i+i)} \right]_{i,j=1}^{n_1} = \det \left[e_{\lambda'_i - i + n_2 + 1 - (n_2+1-j)} \right]_{i,j=1}^{n_2}$$

Remark: If you are not sure whether the signs are correct throughout the proof, ignore them and instead compare leading coefficients:

$S_\lambda = m_\lambda + \dots$ $\qquad$ $\det\left[e_{\lambda'_i - i + j}\right] = m_\lambda + \dots$ (compare with Lecture 2)

Corollary: $\omega(S_\lambda) = S_{\lambda'}$

Proof $\omega\left(\det\left[e_{\lambda'_i - i + j}\right]\right) = \det\left[h_{\lambda'_i - i + j}\right] = S_{\lambda'}$. $\blacksquare$

Corollary (dual Cauchy):

$$\sum_{\lambda \in Y} S_\lambda(x_1, x_2, \dots) S_{\lambda'}(y_1, y_2, \dots) = \prod_{i,j \geq 1} (1 + x_i y_j)$$

Proof: $\sum S_\lambda(x_1, \dots) S_\lambda(y_1, \dots) = \prod_{i,j} \frac{1}{(1 - x_i y_j)} = \prod_i \left(\sum_k h_k(y_1, \dots)(x_i)^k\right)$

Apply ω in $y_1, y_2 \dots$: $\sum S_\lambda(x_1, \dots) S_{\lambda'}(y_1, \dots) = \prod_i \left(\sum_k e_k(y_1, \dots)(x_i)^k\right) = \prod_{i,j} (1 + x_i y_j)$ $\blacksquare$

How about expansion of s_λ in products of p_k?

This turns out to be related to representation theory of symmetric groups S_n

We present an overview without proofs.

See [Section 7, Chapter I, Macdonald's book] for a detailed exposition

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S_n = group of all permutations of $\{1,2,\dots,n\}$ (= bijections from $\{1,..,n\}$ to itself)

Conjugacy classes in S_n are parameterized by partitions of n: two permutations $\sigma, \tau \in S_n$ are conjugate ($\exists a \in S_n : a\sigma a^{-1} = \tau$) iff their cyclic structures are the same.

Ferdinand Georg Frobenius

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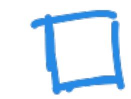
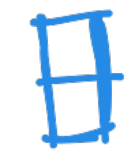
Born	26 October 1849 Charlottenburg , Berlin
Died	3 August 1917 (aged 67) Berlin
Nationality	German
Alma mater	University of Göttingen University of Berlin
Known for	Differential equations Group theory Cayley–Hamilton theorem Frobenius method Frobenius matrix

Scientific career

Fields	Mathematics
Institutions	University of Berlin ETH Zurich
Doctoral advisor	Karl Weierstrass Ernst Kummer
Doctoral students	Richard Fuchs Edmund Landau Issai Schur Konrad Knopp Walter Schnee

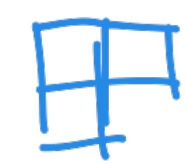
cycles

S_1 : unique permutation (1). 1: 

S_2 : Permutation $12 = Id = (1)(2)$ 2: 

Permutation $21 = (12)$ 1: 

S_3 : Permutation $123 = Id = (1)(2)(3)$ 3: 

Permutations 213, 132, 321 2: 

[transpositions] $(12)(3)$ $(1)(23)$ $(13)(2)$

$\boxed{|S_n| = n!}$

Permutations 231, 312 1: 

[3-cycles] (123) (132)

Notations: one row notation $\sigma \in S_n$ — $\sigma(1)\sigma(2)\dots\sigma(n)$

cycle $(i_1\, i_2 \dots i_k)$ means $i_1 \xrightarrow{\sigma} i_2 \xrightarrow{\sigma} \dots \xrightarrow{\sigma} i_k$

Theorem: in any finite group the number of distinct (non-isomorphic) irreducible representations is the same as the number of conjugacy classes.
(we do not give a proof)

Corollary: Irreducible representations of S_n are parameterized by partitions of n = Young diagrams with n boxes.

$\pi^\lambda : S_n \to GL(V_\lambda)$ irreducible representation corresponding to $\lambda \in Y_n$

homomorphism

vector space of representation

$\chi^\lambda : S_n \to \mathbb{C}$ $\quad \chi^\lambda(\sigma) = \mathrm{Trace}(\pi^\lambda(\sigma))$ character of π^λ

[χ^λ is central = conjugation-invariant]

χ^λ_ρ = value of χ^λ on $\sigma \in S_n$ from conjugacy class $\rho \in Y_n$

Theorem: (we do not give a proof)

[existence and irreducibility of π^λ - part of the theorem]

$$S_\lambda = \sum_{\rho \in Y_{|\lambda|}} \chi^\lambda_\rho \frac{P_\rho}{z_\rho} \qquad (*)$$

$$P_\rho = \sum_{\lambda \in Y_{|\rho|}} \chi^\lambda_\rho S_\lambda \qquad (**)$$

where $z_\rho = \prod_{i \geq 1} i^{m_i} (m_i)!$ $\qquad (0! = 1)$

$m_i = m_i(\rho) = \#$ parts in ρ equal to i

Remark: See next lecture for the equivalence $(*) \Leftrightarrow (**)$

In words, the character table of S_n is given by transition matrices between linear bases $\{S_\lambda\}_{\lambda \in Y}$ and $\{P_\rho\}_{\rho \in Y}$.

But how do you actually compute them?

Example: S_2 has two irreducible representations, both are <u>one-dimensional</u>.

$\pi^{\square\square}$ – trivial representation: both 12 and 21 act as Id

$\pi^{\boxminus}$ – sign/alternating representation: $\pi(12) = \mathrm{Id}$, $\pi(21) = -\mathrm{Id}$

$$\chi^{\square\square}_{\boxminus} = \chi^{\square\square}_{\square\square} = 1 \quad ; \quad \chi^{\boxminus}_{\boxminus} = 1 \quad ; \quad \chi^{\boxminus}_{\square\square} = -1$$

$$S_{\square\square} = h_2 = \sum_i x_i^2 + \sum_{i<j} x_i x_j = \frac{1}{2} p_{\square\square} + \frac{1}{2} p_{\boxminus}$$

$$\frac{1}{2} p_{\square\square} = \frac{1}{2} p_2 = \frac{1}{z_{\square\square}} \chi^{\square\square}_{\square\square} \cdot p_{\square\square}, \qquad \frac{1}{2} p_{\boxminus} = \frac{1}{2} (p_1)^2 = \frac{1}{z_{\boxminus}} \chi^{\square\square}_{\boxminus} p_{\boxminus}$$

it WORKS!

$$z_{\square\square} = 2^1 \cdot 1! = 2$$

$$z_{\boxminus} = 1^2 \cdot 2! = 2$$

$$S_{\boxminus} = e_2 = \sum_{i<j} x_i x_j = -\frac{1}{2} p_{\square\square} + \frac{1}{2} p_{\boxminus} = \frac{1}{z_{\square\square}} \chi^{\boxminus}_{\square\square} p_{\square\square} + \frac{1}{2} \chi^{\boxminus}_{\boxminus} p_{\boxminus}$$

Theorem: $\dim\lambda = \dim(V_\lambda) = \chi^\lambda(12..n) = \chi^\lambda_{1^n} =$

dimension of representation

$\in \mathbb{Y}_n$

$$= \frac{n!}{\prod_{i=1}^{N}(\lambda_i+N-i)!} \cdot \prod_{1\le i<j\le N}(\lambda_i - i - \lambda_j + j),$$

where $N \geqslant \ell(\lambda)$ [can be arbitrary]

Proof: Set $x_1 = .. = x_M = \frac{1}{M}$, $x_{M+1} = x_{M+2} = .. = 0$

$$S_\lambda\left(\tfrac{1}{M}, -, \tfrac{1}{M}\right) = \sum_\rho \chi^\lambda_\rho \, p_\rho\left(\tfrac{1}{M}, .. \tfrac{1}{M}\right) \cdot z_\rho^{-1} \qquad (***)$$

$$p_1\left(\tfrac{1}{M}, -- \tfrac{1}{M}\right) = 1. \qquad p_k\left(\tfrac{1}{M}, .. \tfrac{1}{M}\right) = M^{1-k} \xrightarrow[\text{if } k>1]{M\to\infty} 0$$

Sending $M \to \infty$ in $(***)$ right-hand side becomes

$$\chi^\lambda_{1^n} \cdot z^{-1}_{1^n} = \frac{1}{n!}\chi^\lambda_{1^n}$$

Hence,

$$\frac{\dim\lambda}{n!} = \lim_{M\to\infty} s_\lambda(1^M)\cdot M^{-|\lambda|} = \lim_{M\to\infty}\left[\prod_{1\le i<j\le M}\frac{(\lambda_i-i)-(\lambda_j-j)}{j-i}\right]M^{-|\lambda|}$$

Take $N \geq \ell(\lambda)$ and split the product

$$\prod_{1\le i<j\le N}\left[(\lambda_i-i)-(\lambda_j-j)\right] \cdot \underbrace{\prod_{i=1}^{N}\prod_{j=N+1}^{M}\frac{\lambda_i-i+j}{j-i} \cdot \frac{M^{-\lambda}}{1!\,2!\cdot\ldots\cdot(N-1)!}}$$

$$\prod_{i=1}^{N}\frac{(\lambda_i-i+M)!}{(-i+M)!}\cdot\frac{\cancel{(N-i)!}}{(\lambda_i-i+N)!}\cdot\frac{M^{-\lambda}}{\cancel{1!\,2!\cdot\ldots\cdot(N-1)!}}$$

$$\prod_{i=1}^{N}\frac{1}{(\lambda_i-i+N)!}\cdot\frac{M-i+1}{M}\cdot\ldots\cdot\frac{M-i+\lambda_i}{M}$$

$$\Big\downarrow\; M\to\infty$$

$$\prod_{i=1}^{N}\frac{1}{(\lambda_i-i+N)!}$$

$\blacksquare$

Theorem: $\dim \lambda = \#$ Standard Young Tableaux of shape λ

$= \#$ fillings of boxes of λ with $\{1, \dots, n\}$, strictly increasing in rows/columns

$\mathbb{N}$

$$\begin{array}{|c|c|c|}\hline 1 & 2 & 6 \\ \hline 3 & 5 & \\ \cline{1-2} 4 & & \\ \cline{1-1} 7 & & \\ \cline{1-1}\end{array}$$

Example of SYT

Proof:
$$p_1 s_\lambda = \sum_{\mu = \lambda + \square} s_\mu \qquad \text{(See HW2)}$$

Hence,
$$p_1^n = \sum_{\lambda \in Y_n} s_\lambda \cdot \#\text{of ways to create } \lambda \text{ by adding boxes one after another}$$

$\updownarrow$ same things

$$= \sum_{\lambda \in Y_n} s_\lambda \cdot \#\,\text{SYT of shape } \lambda \qquad \blacksquare$$

Exercise: Give another proof of the theorem by passing $M \to \infty$ in the combinatorial formula for $s_\lambda\left(\frac{1}{M}, \dots, \frac{1}{M}\right)$.

For general λ and ρ, the formula for χ^λ_ρ is much less explicit:

Murnaghan–Nakayama rule

From Wikipedia, the free encyclopedia

Theorem:

$$\chi^\lambda_\rho = \sum_{T \in BST(\lambda,\rho)} (-1)^{ht(T)}$$

where the sum is taken over the set BST(λ,ρ) of all *border-strip* tableaux of shape λ and type ρ. That is, each tableau *T* is a tableau such that

- the *k*-th row of *T* has λ_k boxes
- the boxes of *T* are filled with integers, with the integer *i* appearing ρ_i times
- the integers in every row and column are weakly increasing
- the set of squares filled with the integer *i* form a *border strip*, that is, a connected skew-shape with no 2×2-square.

The *height*, *ht*(T), is the sum of the heights of the border strips in *T*. The height of a border strip is one less than the number of rows it touches.

It follows from this theorem that the character values of a symmetric group are integers.

For the proof based on expansion

$$s_\lambda p_k = \sum_\mu (?) \cdot s_\mu$$

see [Example 5, Section 7, Chapter 1, Macdonald's book]