Math 740 Skew Schur functions Lecture 10

$S_\mu \cdot S_\nu$ can be expanded in S_λ

$$S_\mu S_\nu = \sum_\lambda \underset{\uparrow}{C^\lambda_{\mu\nu}} S_\lambda$$

Littlewood - Richardson coefficients

So far we know:

1) $C^\lambda_{\mu\nu} \in \mathbb{Z}$ because $s_\mu s_\nu$ is an integral linear combination of m_ρ, which are linked to S_λ by integral unitriangular linear transformation

2) $C^\lambda_{\mu\nu} = 0$ unless $|\lambda| = |\mu| + |\nu|$ comparing degrees

3*) $C^\lambda_{\mu\nu} \geq 0$ using match of s_λ with characters of $\mathcal{U}(N)$

multiplicities in tensor product $T_\mu \otimes T_\nu = \bigoplus C^\lambda_{\mu\nu} T_\lambda$ $\mu, \nu, \lambda \in GT_N$

Definition: For $\lambda, \mu \in Y$, skew Schur function is

$$s_{\lambda/\mu} := \sum_{\nu} c^{\lambda}_{\mu\nu} s_{\nu}$$

Equivalently, they are fixed by identities

$$\langle s_{\lambda/\mu}, s_\nu \rangle = \langle s_\lambda, s_\mu s_\nu \rangle$$

- $s_{\lambda/\mu}$ is a homogeneous symmetric function of degree
$$\deg(s_{\lambda/\mu}) = |\lambda| - |\mu|$$
- $s_{\lambda/(0,\dots)} = s_\lambda$

Theorem (generalized Jacobi-Trudi)

$$s_{\lambda/\mu} = \det\left[h_{(\lambda_i - i)-(\mu_j - j)}\right]_{i,j=1}^{N}, \quad N \geq \ell(\lambda)$$

Proof: Take two sets of variables: $(x_1, x_2, \dots)$ and $(y_1, y_2, \dots)$

$$\sum_\lambda s_{\lambda/\mu}(x_1,\dots)\, s_\lambda(y_1,\dots) \overset{\substack{\text{def of } s_{\lambda/\mu}\\ \downarrow}}{=} \sum_{\lambda,\nu} c^\lambda_{\mu\nu}\, s_\nu(x_1,\dots)\, s_\lambda(y_1,\dots) \overset{\substack{\text{def of } c^\lambda_{\mu\nu}\\ \downarrow}}{=}$$

$$= \sum_\nu s_\nu(x_1,\dots)\, s_\mu(y_1,\dots)\, s_\nu(y_1,\dots) \overset{\substack{\text{Cauchy-Littlewood}\\ \downarrow}}{=} s_\mu(y_1,\dots) \sum_\nu h_\nu(x_1,\dots)\, m_\nu(y_1,\dots)$$

Now restrict to large, but finite set of variables $(y_1, \dots y_N)$

$s_{\lambda/\mu}$ is a coefficient of $s_\lambda(y_1,\dots y_n)$ in the expansion of RHS.

By general principle, we multiply by $\prod_{i<j}(y_i - y_j)$ to compute it

$s_{\lambda/\mu}(x_1,\dots) =$ Coefficient of $y_1^{\lambda_1+N-1}\cdot\dots\cdot y_N^{\lambda_N}$ in

$$\sum_{\nu_1\geq 0,\, \nu_2\geq 0,\dots \nu_N\geq 0} h_{\nu_1}\cdot\dots\cdot h_{\nu_N}(x_1,\dots) \bullet y_1^{\nu_1}\cdot\dots\cdot y_N^{\nu_N} \cdot \sum_\sigma (-1)^\sigma y_1^{\mu_{\sigma(1)}+N-\sigma(1)}\cdot\dots\cdot y_N^{\mu_{\sigma(N)}+N-\sigma(N)}$$

$$= \sum_{\sigma\in S_N} (-1)^\sigma h_{(\lambda_1-1)-(\mu_{\sigma(1)}-\sigma(1))}\cdot\dots\cdot h_{(\lambda_N-N)-(\mu_{\sigma(N)}-\sigma(N))} \qquad \blacksquare$$

Corollary: We also have

$$s_{\lambda/\mu} = \det\left[e_{\lambda'_i - i - (\mu'_j - j)}\right]_{i,j=1}^{N} \quad , N \geq \lambda_1$$

Proof: This follows from $HE = \mathrm{Id}$ for $H = [h_{j-i}]$, $E = [(-1)^{j-i} e_{j-i}]$, as in Lecture 7 ■

Corollary: $\omega(s_{\lambda/\mu}) = s_{\lambda'/\mu'}$

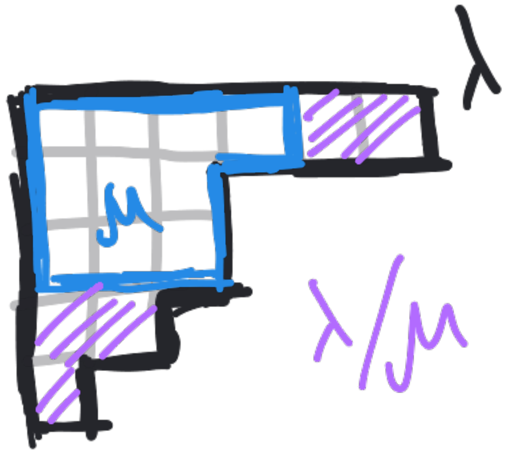
Proof: $\omega\left(\det\left[h_{\lambda_i - i - \mu_j + j}\right]\right) = \det\left[e_{\lambda_i - i - \mu_j + j}\right]$ ■

Theorem: $s_{\lambda/\mu} = 0$ unless $\mu \subset \lambda$, which means $\mu_i \leq \lambda_i$.

Proof: If $\mu_K > \lambda_K$, then $(\lambda_i - i) - (\mu_j - j) < 0$ for $i \geq K$, $j \leq K$

Hence, $h_{(\lambda_i - i) - (\mu_j - j)}$ has $(N-K+1) \times K$ corner of zeros and its determinant vanishes ■

Proposition Draw λ/μ as a skew Young diagram — — set-theoretical difference of λ and μ.



If λ/μ splits into connected components Θ_i, then $S_{\lambda/\mu}$ factorizes

$$S_{\lambda/\mu} = \prod_i S_{\Theta_i}$$

Proof. Suppose that one connected component of λ/μ is in the first r rows and another one in rows $> r$

Then $\mu_r \geq \lambda_{r+1}$ (otherwise there adjacent boxes of λ/μ in rows r and $r+1$)

Thus, $\left[h_{(\lambda_i-i)-(\mu_j-j)}\right]$ has $(N-r)\times r$ corner of zeros

$$\left[h_{(\lambda_i-i)-(\mu_j-j)}\right] = \begin{array}{r} \\ N-r \end{array}\begin{bmatrix} A & B \\ \underset{r}{0} & C \end{bmatrix}$$

$$\det\left[h_{\lambda_i-i-(\mu_j-j)}\right]_{i,j=1}^{N} = \det A \cdot \det C$$

∎

Proposition: Plug in n variables $(x_1,\dots x_n)$

$s_{\lambda/\mu}(x_1,\dots x_n) = 0$ unless $\lambda'_i - \mu'_i \le n$ for all i

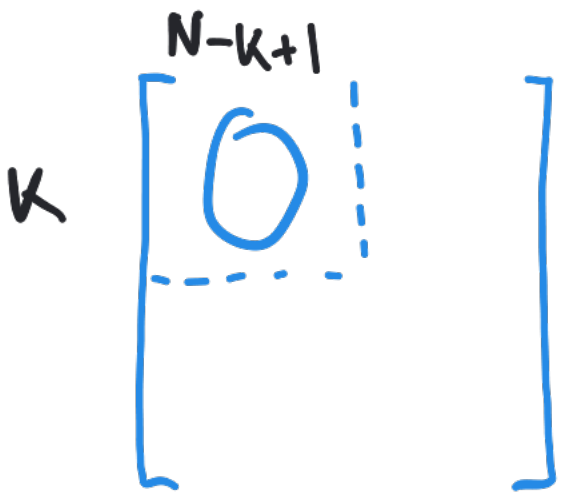
[„height" of λ/μ is $\le n$ in each column]

Proof $s_{\lambda/\mu} = \det\left[e_{(\lambda'_i-i)-(\mu'_j-j)}\right]_{i,j=1}^{N}$

When we have n variables, $e_K \to 0$, $K > n$.

If $\lambda'_K - \mu'_K > n$, then $\lambda'_i - i - (\mu'_j - j) > n$ for $i \le K$, $j \ge K$

Hence, $\left[e_{(\lambda'_i-i)-(\mu'_j-j)}(x_1,\dots x_N)\right]$ has $K \times (N-K+1)$ corner of 0



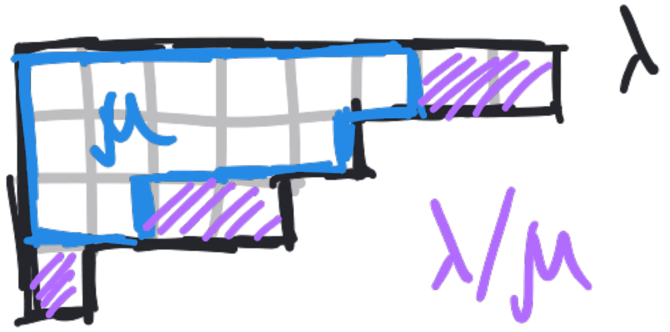
$\Rightarrow$ determinant vanishes.

∎

Corollary: Take one variable

$$s_{\lambda/\mu}(x) = \begin{cases} x^{|\lambda|-|\mu|}, & \lambda \succ \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Proof: $\lambda \succ \mu$ means that they differ by a horizontal strip = collection of boxes with ≤1 in each column.



Hence, by the previous proposition only for such λ, μ, $s_{\lambda/\mu}(x) \neq 0$.

When it is non-zero, it is a homogeneous polynomial in x of degree $|\lambda|-|\mu|$ and with leading coefficient 1.

Thus, it is $x^{|\lambda|-|\mu|}$ $\blacksquare$

Remark: Once we know that λ/μ is a horizontal strip, we can also use factorization from slide 5 to compute it.

Next task: To develop a more abstact version of the combinatorial formula for Schur functions which also works for skew Schur functions

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In the branching rule for s_λ we were splitting $(x_1, \dots x_N)$ into two groups $(x_1, \dots x_{N-1})$, (x_N)

In Λ we can also split our countable collection of variables into two groups.

Formally, we have a co-product: map $\Lambda \to \Lambda \otimes \Lambda$ given on generators by $p_k \longrightarrow p_k(x_1, x_2, \dots) + p_k(y_1, y_2, \dots)$

Interpretation is that we split the variables into two countable groups (say, odd/even indices), call the first $\{x_i\}$, the second $\{y_i\}$

What is happening with (skew) Schur functions under split?

Theorem: $s_{\lambda/\mu}(x_1, x_2, \dots, y_1, y_2, \dots) = \sum_{\nu} s_{\lambda/\nu}(x_1, \dots)\, s_{\nu/\mu}(y_1, \dots)$

[generalized branching rule]

Proof: We first deal with $\mu = (0)$ case $(s_{\lambda/(0)} = s_\lambda)$

Take three sets of variables $(x_1, \dots)$, $(y_1, \dots)$, $(z_1, \dots)$

$$\sum_{\lambda,\mu} s_{\lambda/\mu}(x_1, \dots) s_\lambda(z_1, \dots) s_\mu(y_1, \dots) = \sum_{\lambda,\mu,\nu} c^{\lambda}_{\mu\nu} s_\nu(x_1, \dots) s_\lambda(z_1, \dots) s_\mu(y_1, \dots) =$$

$$= \sum_{\mu,\nu} s_\nu(x_1, \dots) s_\mu(z_1, \dots) s_\nu(z_1, \dots) s_\mu(y_1, \dots) \overset{\text{Cauchy-Littlewood}}{=} \prod_{i,j} \frac{1}{1 - y_i z_j} \cdot \prod_{i,j} \frac{1}{1 - x_i z_j}$$

$$\overset{\text{Cauchy-Littlewood again}}{=} \sum_{\lambda} s_\lambda(x_1, \dots, y_1, \dots) \cdot s_\lambda(z_1, \dots)$$

Comparing the coefficient of $s_\lambda(z_1, \dots)$ in both sides of the identity we get the result.

Switching to the general case:

$$\sum_{\mu} s_{\lambda/\mu}(x_1,\dots,y_1,\dots)\, s_\mu(z_1,\dots) \overset{\text{just proved}}{=} s_\lambda(x_1,\dots,y_1,\dots,z_1,\dots) =$$

$$\underset{\text{just proved}}{=} \sum_{\nu} s_{\lambda/\nu}(x_1,\dots)\, s_\nu(y_1,\dots,z_1,\dots) \overset{\text{just proved}}{=} \sum_{\nu,\mu} s_{\lambda/\nu}(x_1,\dots)\, s_{\nu/\mu}(y_1,\dots)\cdot$$

$$\cdot\, s_\mu(z_1,\dots)$$

Comparing the coefficient of $s_\mu(z_1,\dots)$ we get the desired statement. ▨

Corollary: $s_\lambda(x_1,\dots,y_1,\dots) = \sum_{\mu,\nu} c^{\lambda}_{\mu,\nu}\, s_\mu(x_1,\dots)\, s_\nu(y_1,\dots)$

[Hence, we can find Littlewood-Richardson coef. from this expansion]

Proof. Both sides are equal to

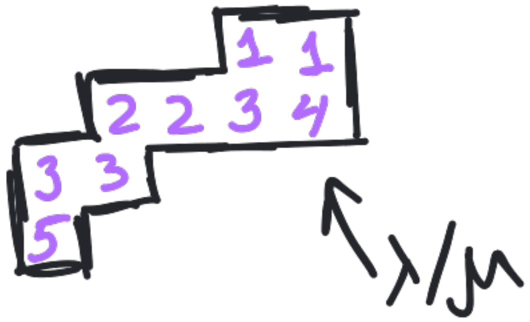
$$\sum_{\mu} s_{\lambda/\mu}(y_1,\dots)\, s_\mu(x_1,\dots)$$

▨

Corollary: $s_{\lambda/\mu}(x_1, x_2, \_ \, x_N) =$

$$= \sum_{\lambda=\nu^N \succ \nu^{N-1} \succ \dots \succ \nu^0=\mu} x_1^{|\nu^1|-|\nu^0|} \cdot \dots \cdot x_N^{|\nu^N|-|\nu^{N-1}|} =$$

$$= \sum_{\text{SSYT of shape } \lambda/\mu} x_1^{\#1} \cdot x_2^{\#2} \cdot \dots \cdot x_N^{\#N}$$



filling of boxes of λ/μ with numbers from $\{1, .. N\}$, weakly growing in rows and strictly in columns.

Proof: Apply the theorem N times. Set x_1 to be the first set of variables,.. , x_N to be the Nth.
Use corollary from slide 7 for $s_{\nu^i/\nu^{i-1}}(x_i)$ ∎

This is the combinatorial formula for $s_{\lambda/\mu}$.

Can take it as an alternative definition.
(symmetry?)