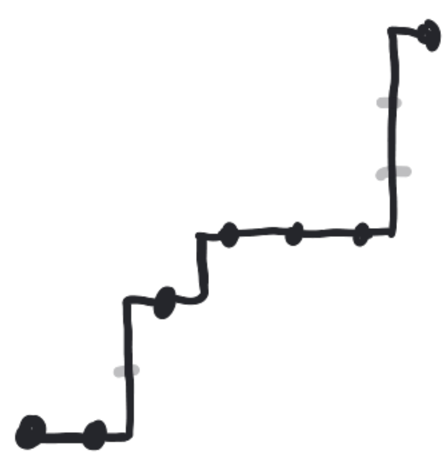
Math 740 Skew Schur functions 2 Lecture 11

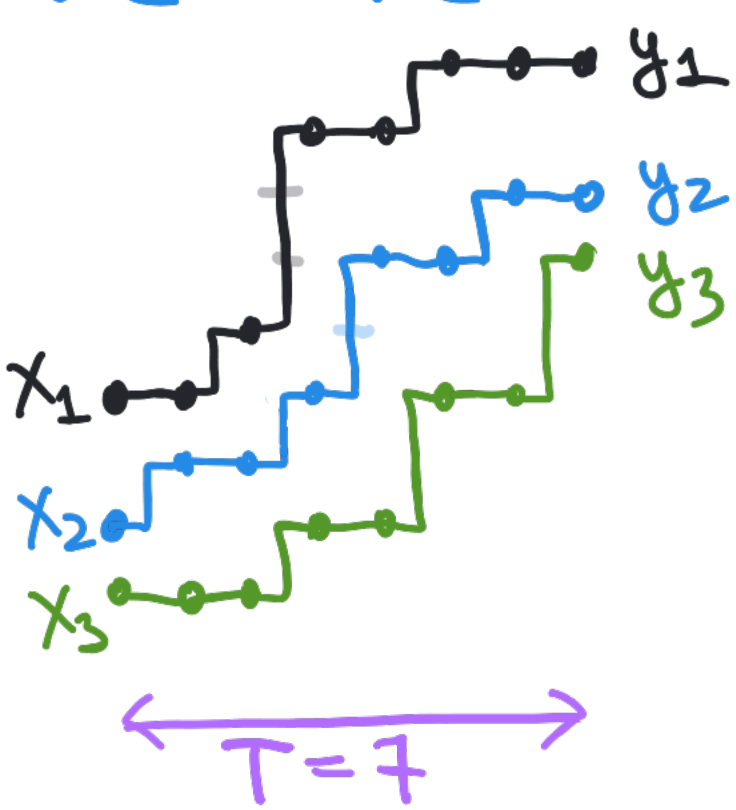
Today we concentrate on combinatorial aspects.

Consider walks of the following kind:



on each step we move $\frac{1}{2}$ to the right, then $k = 0, 1, 2, ..$ up and then another $\frac{1}{2}$ to the right

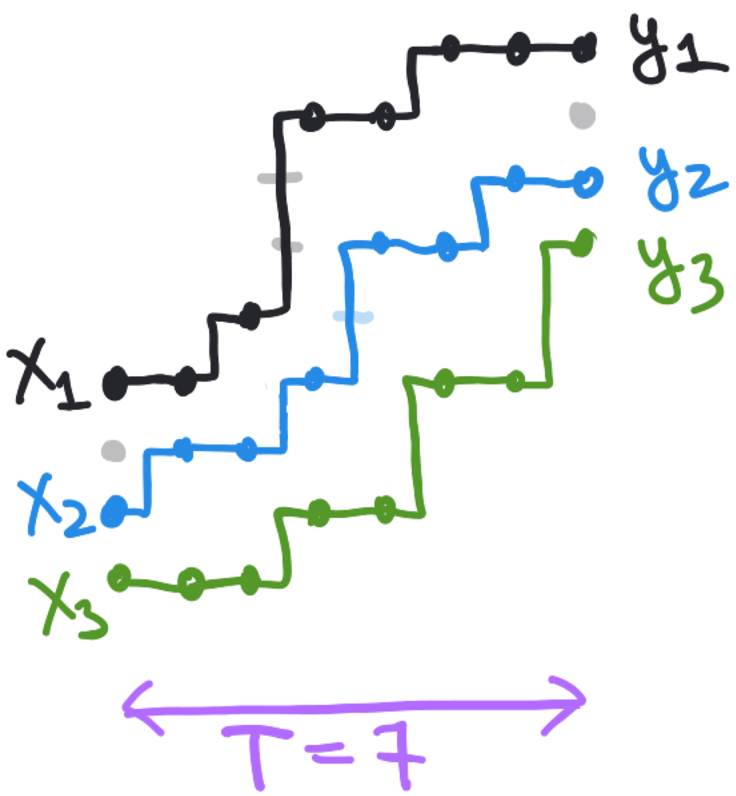
We are interested in non-intersecting families



(they do not intersect and do not touch each other)

Fix n, starting points $x_1 > .. > x_n$, ending points $y_1 > .. > y_n$, number of steps T.

How many non-intersecting families are there?



$\nu^0=(1,0,0)$

$\nu^1=(1,1,0)$

$\nu^2=(2,1,0)$

$\nu^3=(5,2,1)$

Theorem 1: Given n, T, $x_1,\dots x_n$, $y_1,\dots y_n$, the total number of families of non-intersecting paths is:

$$\det\left[\binom{T-1+y_i-x_j}{y_i-x_j}\right]_{i,j=1}^{n}$$

binomial coefficient (vanishes when $y_i-x_j<0$)

Proof: Set $\mu\in\mathbb{Y}$: $X_i=\mu_i+(n-i)$, $i=1,2,\dots n$

$\lambda\in\mathbb{Y}$: $y_i=\lambda_i+(n-i)$, $i=1,2,\dots n$

$\nu^K\in\mathbb{Y}$: $\nu^K_i+(n-i)$ — positions of particles after K steps

$K=0,1,\dots T$. $\quad \nu^0=\mu$, $\nu^T=\lambda$

$\nu^{k} \succ \nu^{k-1}$ means that

$$\nu^{k}_{1}+(n-1) \geq \nu^{k-1}_{1}+(n-1) > \nu^{k}_{2}+(n-2) \geq \nu^{k-1}_{2}+(n-2) > \ldots$$

weak — strict — weak — strict

This is precisely the non-intersecting condition!

Hence, by the combinatorial formula for $s_{\lambda/\mu}$ (last slide Lecture 10)

$\#$ families of paths $= s_{\lambda/\mu}(1^T)$

By the generalized Jacobi-Trudi formula (slide 2 Lecture 10)

$$s_{\lambda/\mu}(1^T) = \det\left[h_{y_i - x_j}(1^T)\right]_{i,j=1}^{n}$$

remains to compute $h_k(1^T)$

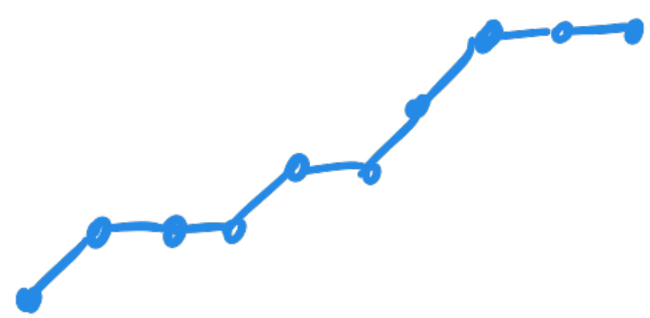
H(t) from Lec 3

$$\sum_{k\geq 0} h_k(1^T)\, t^k = (1-t)^{-T} = \sum_{k\geq 0} \binom{T+k-1}{k} t^k$$

negative Binomial theorem (or Taylor expansion)

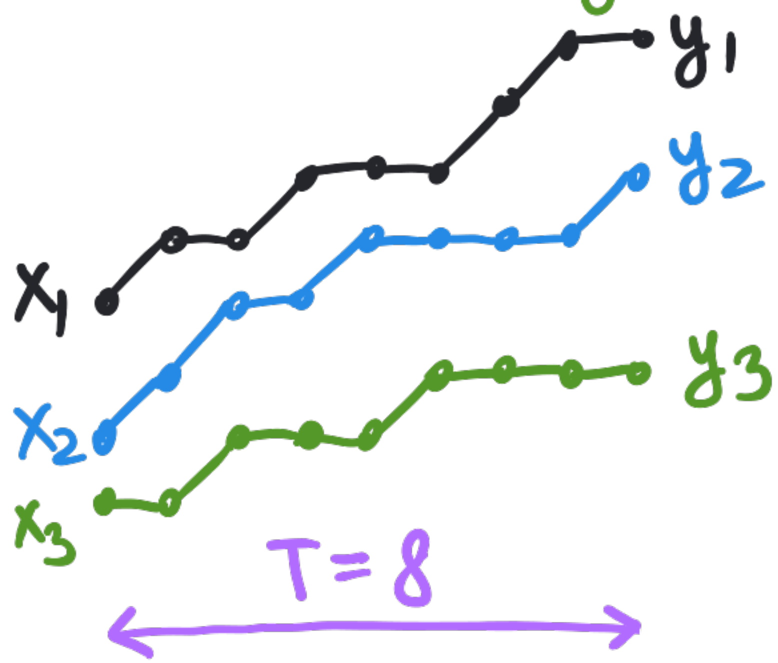
$$h_k(1^T) = \binom{T+k-1}{k}$$

Let's now switch to a different type of walks:



Each step is either  or 

We are again interested in non-intersecting families



(paths are not allowed to share any vertices)

Theorem 2: Given n, T, $x_1 > \dots > x_n$, $y_1 > \dots > y_n$, the total number of families of non-intersecting paths is

$$\det\left[\binom{T}{y_i - x_j}\right]_{i,j=1}^{n}$$

Proof: Apply generalized branching rule (slide 9 in Lecture 10) T times and then specialize each set of variables by the same specialization ρ with $\beta_1 = 1$, all other parameters $=0$.
(Example 2 of Lecture 9) [see slide 3 there]

We get:

$$S_{\lambda/\mu}(\beta_1 = \dots = \beta_T = 1) = \sum_{\mu = \nu^0, \nu^1, \dots \nu^T = \lambda} \prod_{k=1}^{T} S_{\nu^k/\nu^{k-1}}(\beta_1 = 1)$$

this is composition of ω and pluggin in T variables equal to 1

this is composition of ω and plugging in one variable equal to 1.

By corollaries on slides 4 and 7 of Lecture 10

$$S_{\nu^k/\nu^{k-1}}(\beta_1 = 1) = \begin{cases} 1, & \nu_i^k - \nu_i^{k-1} \in \{0,1\} \text{ for all } i, \\ 0, & \text{otherwise.} \end{cases}$$

Hence, matching $\nu_i^k + (n-i)$, $i=1,\dots k$, with particles after k steps $\left(\begin{array}{l} \nu_i^0 + (n-i) = x_i \\ \nu_i^T + (n-i) = y_i \end{array}\right)$ we conclude that

$\#$ families of non-intersecting paths $=$

$$= s_{\lambda/\mu}(\beta_1 = \dots = \beta_T = 1) \overset{\text{generalized Jacobi-Trudi}}{=} \det\left[h_{y_i - x_j}(\beta_1 = \dots = \beta_T = 1)\right]_{i,j=1}^{N}$$

Remains to compute $h_k(\beta_1 = \beta_2 = \dots = 1)$.

$$\sum_{k\geq 0} h_k(\beta_1 = \dots = \beta_T = 1)\, t^k \overset{w(h_k)=e_k}{=} \sum_{k\geq 0} e_k(1^T)\, t^k \overset{\text{formula for } E(t) \text{ from Lecture 3}}{=} (1+t)^T \overset{\text{binomial theorem}}{=} \sum_{k\geq 0} \binom{T}{k} t^k$$

Thus, $h_k(\beta_1 = \dots = \beta_T = 1) = \binom{T}{k}$ $\blacksquare$

Examples for theorem 2:

$n=1$: 

$\#\text{paths} = \binom{T}{y_1-x_1}$ since y_1-x_1 out of T steps should be up.

$n=2$: 

Theorem claims that:

$\#$pairs of non-intersecting paths $=$

$$= \binom{T}{y_1-x_1}\binom{T}{y_2-x_2} - \binom{T}{y_1-x_2}\binom{T}{y_2-x_1}$$

Let's try to prove this formula directly.

The first term counts all pairs of paths.

Is it true that the second term counts intersecting paths?

Yes!

pairs in 2nd term have to intersect.
<u>But</u> they connect $x_1 \to y_2$, $x_2 \to y_1$

swap after last intersection

Exercise 1: Extend the argument of $n=2$ case to get a direct combinatorial proof of Theorem 2

Exercise 2: Do the same for Theorem 1.

The same idea and a similar result appear often

answer = $n \times n$ determinant of 1d results

when counting families of non-intersecting paths.

Pacific Journal of Mathematics

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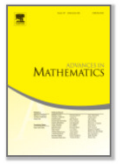
Pacific J. Math.
Volume 9, Number 4 (1959), 1141-1164.

Coincidence probabilities.

Samuel Karlin and James McGregor

ELSEVIER

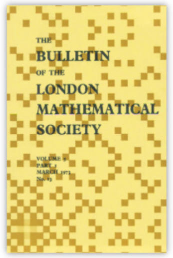
Advances in Mathematics
Volume 58, Issue 3, December 1985, Pages 300-321



Binomial determinants, paths, and hook length formulae

Ira Gessel *, Gérard Viennot

Bulletin of the London Mathematical Society



Volume 5, Issue 1
March 1973
Pages 85-90

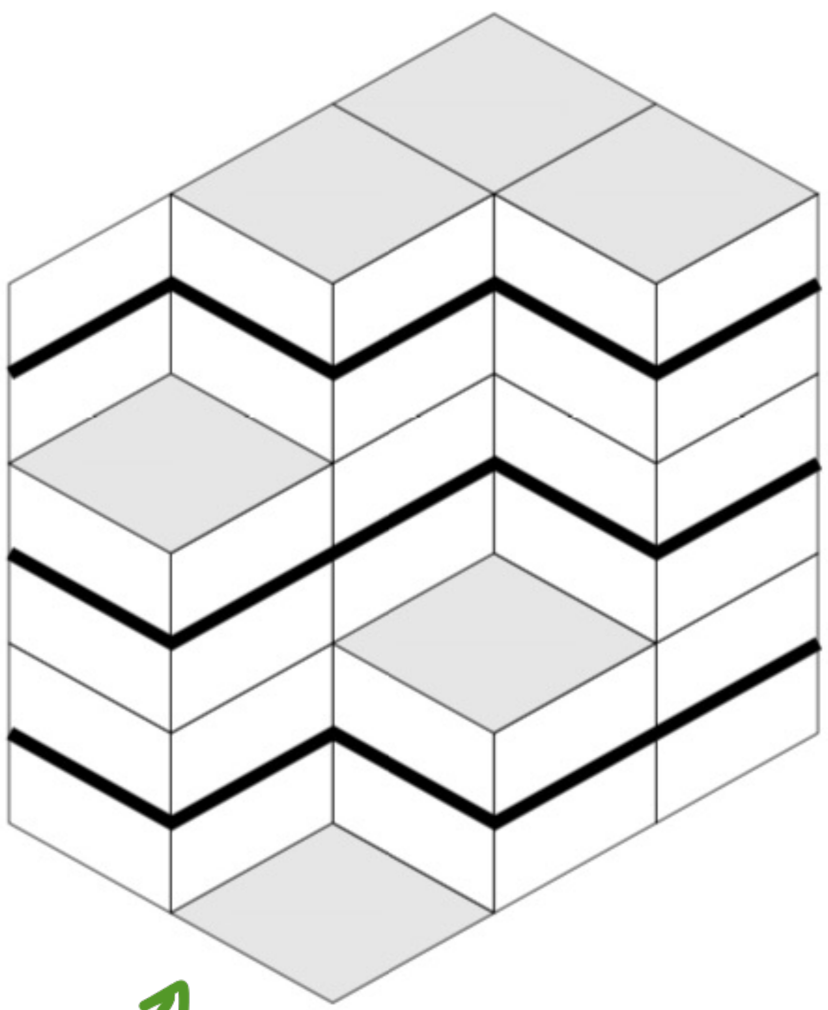
Notes and papers

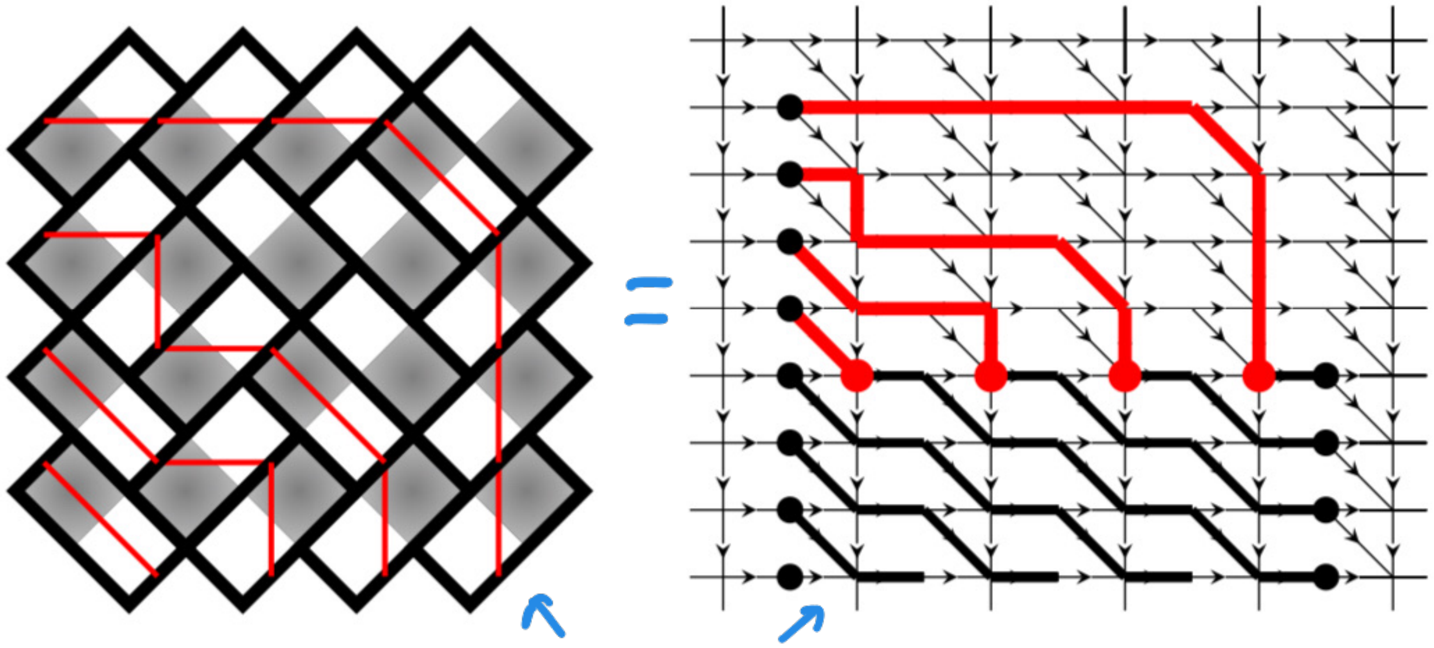
On the Vector Representations of Induced Matroids

Bernt Lindström

Rediscovered several times

Study of families of non-intersecting paths is closely related to the study of tilings





This is

- Tiling of Aztec diamond by 2x1, 1x2 dominos
- Family of non-intersecting paths (hybrid of Th.1 and Th2)

This is

- 3d Young diagram = plane partition
- Family of paths as in Theorem 2
- Lozenge (=rhombus) tiling of 3x3x2 hexagon
(see HW 3 problem 4 for more)

Coming back to $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda$ and $s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu\nu} s_\nu$, there are several combinatorial formulas for $c^\lambda_{\mu\nu}$. We present two.

Archibald Read Richardson

Dudley E. Littlewood

Not J.E. Littlewood!

Group Characters and Algebra

D. E. Littlewood and A. R. Richardson

Philosophical Transactions of the Royal Society of London. Series A of a Mathematical or Physical Character
Vol. 233 (1934), pp. 99-141 (43 pages)
Published By: Royal Society

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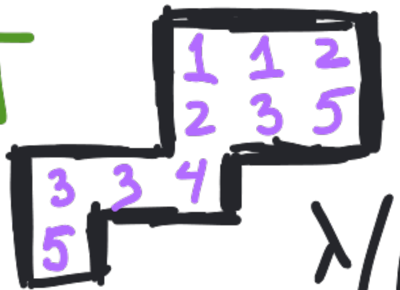


λ/μ

Semistandard Young tableau of shape λ/μ

Read the numbers in each row from right to left.

$w(T) = 2\,1\,1\,5\,3\,2\,4\,3\,3\,5$

like this one

A word of length N in symbols $1, 2, \dots n$ is a lattice permutation if for all $1 \le r \le N$, $1 \le i \le n-1$, # of occurrences of „i" in the first r letters of the word is $\geq$ # of occurences of „$i+1$"

Theorem (Littlewood-Richardson rule)

$c^{\lambda}_{\mu\nu}$ = number of semistandard Young tableaux T of shape λ/μ with weight ν, and such that their words are lattice permutations

as in combinatorial formula (Lecture 4, slide 8)

More history and examples:

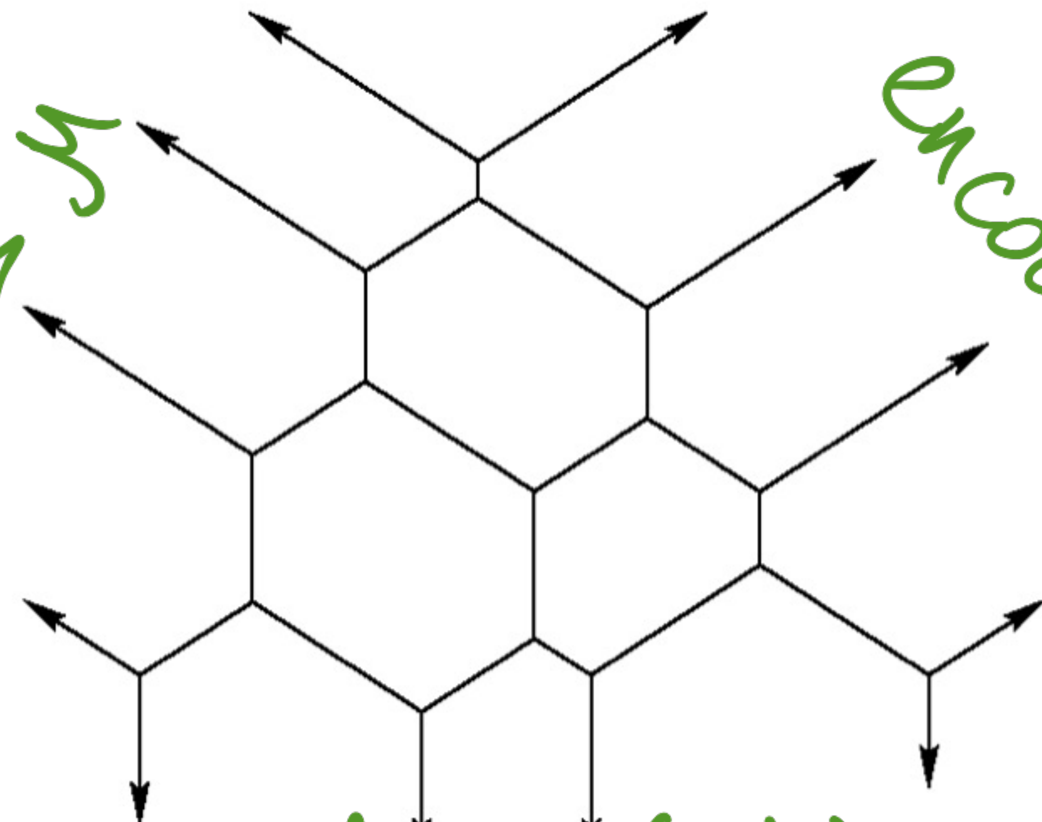
Littlewood–Richardson rule - Wikipedia

In mathematics, the **Littlewood–Richardson rule** is a combinatorial description of the coefficients that arise when decomposing a product of two Schur functions as a linear combination of other S...

en.wikipedia.org/wiki/Littlewood–Richardson_rule

Proof in Macdonald's book: Chapter I, Section 9

A much more recent and beautiful way to compute $c_{\mu\nu}^{\lambda}$ is by enumerating honeycombs



encodes μ

encodes ν

encodes $(-\lambda)$

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AMERICAN MATHEMATICAL SOCIETY
Volume 12, Number 4, Pages 1055–1090
S 0894-0347(99)00299-4
Article electronically published on April 13, 1999

THE HONEYCOMB MODEL OF $GL_n(\mathbb{C})$ TENSOR PRODUCTS I: PROOF OF THE SATURATION CONJECTURE

ALLEN KNUTSON AND TERENCE TAO

Honeycombs and Sums of Hermitian Matrices

Allen Knutson and Terence Tao

(review for Notices of AMS)

There is much more material about $c^{\lambda}_{\mu\nu}$ and skew Schur functions $s_{\lambda/\mu}$, but we do not have time to cover everything.

Examples after each section of Macdonald's book are a good source of further reading.

26. The identities (4.3) and (4.3′) can be generalized as follows: for any two partitions λ, μ we have

$$(1) \qquad \sum_{\rho} s_{\rho/\lambda}(x) s_{\rho/\mu}(y) = \prod_{i,j} (1 - x_i y_j)^{-1} \sum_{\tau} s_{\mu/\tau}(x) s_{\lambda/\tau}(y);$$

$$(2) \qquad \sum_{\rho} s_{\rho/\lambda'}(x) s_{\rho'/\mu}(y) = \prod_{i,j} (1 + x_i y_j) \sum_{\tau} s_{\mu'/\tau}(x) s_{\lambda/\tau'}(y).$$

For instance, here are two versions of skew Cauchy identity.