Math 740 Orthogonality in Λ Lecture 8

Task: to develop $N=\infty$ version of the scalar product on Λ_N of Lecture 5.

Difficulty: It is hard to integrate in infinite-dimensional spaces. Hence, we need another way.

Reminder: $\sum_\lambda S_\lambda(x_1,x_2..)S_\lambda(y_1,y_2,..) = \prod_{i,j\geq 1} \frac{1}{1-x_iy_j}$

We plan to use this formula to define $\langle\cdot,\cdot\rangle$ on Λ

Before we do that, let us develop two more expansions of the right-hand side

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Theorem: $\sum_{\lambda\in\mathbb{Y}} m_\lambda(x_1,..)h_\lambda(y_1,..) = \sum_{\lambda\in\mathbb{Y}} \frac{1}{z_\lambda} p_\lambda(x_1,..)p_\lambda(y_1,..) = \prod_{i,j\geq 1}\frac{1}{1-x_iy_j}$

$\left[z_\lambda = \prod_{i\geq 1} i^{m_i}\cdot m_i! \qquad m_i = \#\text{parts } i \text{ in } \lambda\right]$

Proof: $\displaystyle\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \prod_{i\geq 1} \left(1 + h_1(y_1,\dots)\, x_i + h_2(y_1,\dots)\, x_i^2 + \dots\right)$

Each monomial $x_1^{\lambda_1} \cdot x_2^{\lambda_2} \cdot \dots \cdot x_n^{\lambda_n}$ appears in the expansion of RH exactly once.

Its coefficient is $h_{\lambda_1}(y_1,\dots)\, h_{\lambda_2}(y_1,\dots) \cdot \dots = h_\lambda(y_1,\dots)$

Hence, $\displaystyle\prod \frac{1}{1-x_i y_j} = \sum_\lambda m_\lambda(x_1,x_2,\dots)\, h_\lambda(y_1,y_2,\dots)$

Also $\displaystyle\prod_{i,j\geq 1} \frac{1}{1-x_i y_j} = \prod_{i\geq 1} \exp\left(p_1(y_1,\dots)\, x_i + \frac{1}{2} p_2(y_1,\dots)\, x_i^2 + \dots\right)$

$$= \exp\left(\sum_{k\geq 1} \frac{p_k(y_1,y_2,\dots)\, p_k(x_1,x_2,\dots)}{k}\right)$$

$$= \prod_{k\geq 1}\left(1 + \frac{p_k(y_1,\dots)\, p_k(x_1,\dots)}{k} + \left(\frac{p_k(y_1,\dots)\, p_k(x_1,\dots)}{k}\right)^2 \cdot \frac{1}{2!} + \dots\right)$$

Multiplying, we see the desired $\displaystyle\sum_\lambda \frac{1}{z_\lambda} p_\lambda(x_1,\dots)\, p_\lambda(y_1,\dots)$ ∎

Theorem: The following 4 definitions of scalar product $\langle \cdot, \cdot \rangle$ on Λ are equivalent

1. $\langle h_\lambda, m_\mu \rangle = \langle m_\mu, h_\lambda \rangle = \delta_{\lambda=\mu}$

2. $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda=\mu} \cdot z_\lambda$

3. $\langle s_\lambda, s_\mu \rangle = \delta_{\lambda=\mu}$

4. For any two linear bases of Λ $\{v_\lambda\}, \{u_\lambda\}$, $[\lambda \in Y]$ such that $\sum_{\lambda \in Y} v_\lambda(x_1, \ldots) u_\lambda(y_1, \ldots) = \prod_{i,j \geq 1} \frac{1}{1 - x_i y_j}$, we have $\langle v_\lambda, u_\mu \rangle = \delta_{\lambda=\mu}$.

Proof Take $\{v_\lambda\}$, $\{u_\lambda\}$ as in 4. and define

$$\Big\langle \sum_\lambda \underset{\mathbb{R}}{c_\lambda} v_\lambda , \sum_\lambda \underset{\mathbb{R}}{d_\lambda} u_\lambda \Big\rangle := \sum_{\lambda \in Y} c_\lambda d_\lambda$$

Further, take another pair $\{\tilde{v}_\lambda\}$, $\{\tilde{u}_\lambda\}$ as in 4.

We need to show $\langle \tilde{v}_\lambda , \tilde{u}_\mu \rangle = \delta_{\lambda=\mu}$

For that expand $\tilde{v}_\lambda = \sum_\mu a^\mu_\lambda v_\mu$ $\quad \tilde{u}_\lambda = \sum_\mu b^\mu_\lambda u_\mu$

We know: $\sum_\lambda v_\lambda(x_1,..) u_\lambda(y_1,..) = \sum_\lambda \tilde{v}_\lambda(x_1,..) \tilde{u}_\lambda(x_1,..)$

$$= \sum_{\mu,\rho} v_\mu(x_1,..) u_\rho(y_1,...) \cdot \sum_\lambda a^\mu_\lambda \cdot b^\rho_\lambda$$

Therefore, $\sum_\lambda a^\mu_\lambda \cdot b^\rho_\lambda = \delta_{\mu=\rho}$.

Now we can compute $\langle \tilde{v}_\mu, \hat{u}_\rho \rangle = \langle \sum_\lambda a_\mu^\lambda v_\lambda, \sum_\nu b_\rho^\nu u_\nu \rangle$

by def. of $\langle \cdot, \cdot \rangle$

$$= \sum_\lambda a_\mu^\lambda b_\rho^\lambda = \delta_{\mu=\rho} \quad \text{as desired.}$$

Since we already know that

$$\sum_\lambda s_\lambda(x_1,\dots) s_\lambda(y_1,\dots) = \sum_\lambda h_\lambda(x_1,\dots) m_\lambda(y_1,\dots) = \sum_\lambda m_\lambda(x_1,\dots) h_\lambda(y_1,\dots)$$

$$= \sum_\lambda p_\lambda(x_1,\dots) \cdot \frac{p_\lambda(y_1,\dots)}{z_\lambda} = \sum_\lambda u_\lambda(x_1,\dots) v_\lambda(y_1,\dots),$$

This argument shows the equivalence of 1-4. ∎

Corollary: The involution ω is an isometry, which means $\langle f, g \rangle = \langle \omega(f), \omega(g) \rangle$

Proof. ω maps an orthonormal basis s_λ to an orthonormal basis $s_{\lambda'}$ ∎

Corollary: (alternative definition of Schur functions)

$\{S_\lambda\}_{\lambda \in Y}$ is a unique linear basis in Λ suct that:

1) $S_\lambda = m_\lambda +$ (lower degree terms)

[can use lexicographic or dominance order on Y]

2) $\langle S_\lambda, S_\mu \rangle = 0$ if $\lambda \neq \mu$.

Proof: By Gram-Schmidt orthogonalization procedure such basis is unique. And we already know that Schur functions satisfy these properties. $\blacksquare$

Remark: In order to use this characterization, one should start by defining $\langle \cdot, \cdot \rangle$ either through

$\langle h_\lambda, m_\mu \rangle = \delta_{\lambda=\mu}$ or $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda=\mu} z_\lambda$

Corollary: If we define χ^λ_ρ and $\widetilde{\chi}^\lambda_\rho$ through expansions

$$\left.\begin{aligned} S_\lambda &= \sum_\rho \chi^\lambda_\rho \frac{P_\rho}{z_\rho} \\ P_\rho &= \sum_\lambda \widetilde{\chi}^\lambda_\rho S_\lambda \end{aligned}\right] \text{ then } \chi^\lambda_\rho = \widetilde{\chi}^\lambda_\rho$$

Proof Scalar product of 1st identity with P_ν

gives $\langle S_\lambda, P_\nu \rangle = \chi^\lambda_\nu$

Scalar product of 2nd identity with S_μ

gives $\langle S_\mu, P_\rho \rangle = \widetilde{\chi}^\mu_\rho$ ∎

Reminder: In Lecture 7 we discussed that χ^λ_ρ is the value of character of irred. rep λ of S_n on conjugacy class ρ.