Math 740 $\quad s_\lambda$ as an element of Λ $\quad$ Lecture 6

Task: study Schur functions of infinitely-many variables

Lemma: Take $\lambda \in GT_N \cap \mathbb{Y}$ (means $\lambda_N \geq 0$)

$$s_\lambda(x_1,\ldots,x_{N-1},0) = \begin{cases} 0, & \lambda_N > 0, \\ s_{\lambda}(x_1,\ldots x_{N-1}), & \lambda_N = 0. \end{cases}$$

last coordinate removed

Proof: Use combinatorial formula:

$$s_\lambda(x_1,\ldots x_{N-1},x) = \sum_{\mu \prec \lambda} x^{|\lambda|-|\mu|} s_\mu(x_1,\ldots x_{N-1})$$

When we plug $x=0$, the only surviving term has $\lambda_1=\mu_1$, $\lambda_2=\mu_2, \ldots$, $\lambda_{N-1}=\mu_{N-1}$, $\lambda_N=0$. $\blacksquare$

Def.: Take a partition $\lambda \in \mathbb{Y}$

$$\underset{\Lambda}{\overset{\cap}{S_\lambda}} = (\underbrace{0; \dots; 0}_{\substack{\text{when \#variables} \\ < N = \ell(\lambda)}}; \underset{GT_N}{\overset{}{S_\lambda}}(x_1, \dots x_N); \underset{GT_{N+1}}{S_\lambda}(x_1, \dots x_{N+1}), \dots)$$

Proposition: $S_\lambda = \sum_{\mu \leq \lambda} K_\lambda^\mu \, m_\mu$, where

K_λ^μ is the Kostka number — number of semistandard Young tableaux of shape λ, weight μ.

Proof: Send $N \to \infty$ in N-variable statement of Lecture 4. Notice that after $N > |\lambda|$ the expansion stabilizes and does not change anymore. ∎

Corollary: $\{s_\lambda\}_{\lambda\in\mathbb{Y}}$ is a linear basis of Λ

Proof: By proposition it is related to $\{m_\lambda\}_{\lambda\in\mathbb{Y}}$ by a unitriangular transformation. ∎

Next tasks: A) How to expand functions in s_λ?
B) How are s_λ related to e_k, h_k, p_k?

Central expansion of the entire class

Theorem (Cauchy [-Littewood] identity):

$$\sum_{\lambda\in\mathbb{Y}} s_\lambda(x_1,x_2,\dots)\, s_\lambda(y_1,y_2,\dots) = \prod_{i,j\geq 1} \frac{1}{1-x_i y_j}$$

a series (growing degrees!) of elements in $\Lambda_{\vec{x}}$ with coefficients from $\Lambda_{\vec{y}}$

expand $\prod_{i,j}\left(1+(x_iy_j)+(x_iy_j)^2+\dots\right)$ to get a similar series

Proof: Step 1: Suffices to check N-variable version

$$\sum_{\lambda \in \mathbb{Y} \cap GT_N} s_\lambda(x_1, \dots, x_N) s_\lambda(y_1, \dots, y_N) = \prod_{i,j=1}^{N} \frac{1}{1 - x_i y_j}$$

Why? Because in degree K nothing changes as $N \to \infty$ after $N > K$.

General principle: Identity in Λ holds if and only if it holds in Λ_N for all large enough N.

Step 2: Cauchy determinant

$$(*) \qquad \det\left[\frac{1}{x_i - y_j}\right]_{i,j=1}^{N} = \frac{\prod_{i<j} (x_i - x_j)(y_j - y_i)}{\prod_{i,j=1}^{N} (x_i - y_j)}$$

[note $x_{\underline{i}} - x_j$, but $y_{\underline{j}} - y_i$]

Proof of (*): Multiply both sides by $\prod_{i,j}(x_i - y_j)$

Now both sides are polynomials in x_i, y_j of degree $N(N-1)$.

Left-hand side is skew-symmetric in x_i.

(changes sign under $x_i \leftrightarrow x_j$, vanishes at $x_i = x_j$)

Hence, it is divisible by $\prod_{i<j}(x_i - x_j)$ ← degree $\frac{N(N-1)}{2}$

Left-hand side is also skew-symmetric in y_i.

Hence, it is divisible by $\prod_{i<j}(y_i - y_j)$ ← degree $\frac{N(N-1)}{2}$

Comparing degrees we conclude:

$$\det\left(\frac{1}{x_i - y_j}\right) \cdot \prod_{i,j=1}^{N}(x_i - y_j) = (\text{const}) \cdot \prod_{i<j}(x_i - x_j)(y_j - y_i)$$

Computing some monomial in both sides, we see that $(\text{const}) = 1$. ∎

Step 3. Replace y_i by $\frac{1}{y_i}$ and get

$$\det\left[\frac{1}{1-x_i y_j}\right]_{i,j=1}^{N} = \frac{\prod_{i<j}(x_i-x_j)(y_i-y_j)}{\prod_{i,j=1}^{N}(1-x_i y_j)}$$

Next, $\det = \det\left[1+(x_i y_j)+(x_i y_j)^2+(x_i y_j)^3+\dots\right]_{i,j=1}^{N} =$

$$= \det\Big(\underbrace{[x_i^k]}_{N\times\infty \text{ matrix}}\cdot\underbrace{[y_j^k]}_{\infty\times N \text{ matrix}}\Big)$$

Apply Cauchy-Binet formula (see wikipedia if you don't know it)

$$= \sum_{k_1 > \dots > k_N} \det(x_i^{k_j})\cdot\det(y_i^{k_j})$$

Dividing by $\prod_{i<j}(x_i-x_j)(y_i-y_j)$ and identifying $k_i = \lambda_i + N - i$ we recognize Schur polynomials $\blacksquare$

Matches $\sum s_\lambda(x_i)\, s_\lambda(y_i)$!

We now develop $h_\lambda \to s_\lambda$ transition as a corollary.

Theorem (Jacobi-Trudi formula)

$$s_\lambda = \det\left[h_{\lambda_i - i + j}\right]_{i,j=1}^{n}, \text{ where}$$

$h_0 = 1$, $h_k = 0$ for $k < 0$, n - arbitrary number $\geq \ell(\lambda)$

Our proof is based on an auxiliary statement of independent interest

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Proposition: Consider a power series

$$F(t) = 1 + f_1 t + f_2 t^2 + f_3 t^3 + \dots$$

Define c_λ as coefficient of expansion $\prod_{i \geq 1} F(x_i) = \sum_\lambda c_\lambda s_\lambda$

Then $c_\lambda = \det\left[f_{\lambda_i - i + j}\right]_{i,j=1}^{n}$, $f_0 = 1$, $f_k = 0$ for $k < 0$.

Proof of Proposition. It suffices to prove N-variable version

$$\prod_{i=1}^{N} F(x_i) = \sum_{\lambda \in \mathbb{Y} \cap GT_N} c_\lambda \cdot s_\lambda(x_1, \dots x_N) \qquad \left[\begin{array}{l}\text{same formula}\\ \text{for } c_\lambda\end{array}\right]$$

Since s_λ form a linear basis in Λ_N, the expansion of $\prod_{i=1}^{N} F(x_i)$ exists. How to find coefficients?

General principle: In order to expand in $s_\lambda(x_1, \dots x_N)$, multiply by $\prod_{i<j}(x_i - x_j)$. Then c_λ becomes the coefficient of monomial $x_1^{\lambda_1+N-1} \cdot x_2^{\lambda_2+N-2} \cdot \ldots \cdot x_N^{\lambda_N}$

$$\prod_{i=1}^{N} F(x_i) \prod_{i<j}(x_i - x_j) = \prod_{i=1}^{N}\left(1 + f_1 x_i + f_2 (x_i)^2 + \dots\right) \cdot \sum_{\sigma \in S_N} (-1)^\sigma x_{\sigma(1)}^{N-1} \cdot x_{\sigma(2)}^{N-2} \cdot \ldots \cdot x_{\sigma(N-1)}$$

There are $N!$ ways to get a monomial $x_1^{\lambda_1+N-1} \cdot \ldots \cdot x_N^{\lambda_N}$ in this sum by taking

$x_1^{N-\sigma^{-1}(1)} \cdot \ldots \cdot x_N^{N-\sigma^{-1}(N)} \cdot (-1)^{\sigma}$ and multiplying by

$x_1^{\lambda_1-1+\sigma^{-1}(1)} \cdot \ldots \cdot x_N^{\lambda_N-N+\sigma^{-1}(N)} \cdot f_{\lambda_1-1+\sigma^{-1}(1)} \cdot \ldots \cdot f_{\lambda_N-N+\sigma^{-1}(N)}$

Sum of green factors over σ $= \det\left[f_{\lambda_i-i+j}\right]_{i,j=1}^{N}$ $\blacksquare$

Exercise: Let $F(t) = \sum_{n \in \mathbb{Z}} f_n t^n$ [Negative n allowed!]

Then still $\prod_{i=1}^{N} F(x_i) = \sum_{\lambda \in GT_N} c_\lambda \, s_\lambda(x_1, \ldots x_N)$

admits a formula $c_\lambda = \det\left[f_{\lambda_i-i+j}\right]_{i,j=1}^{N}$

[Now all f_n might be $\neq 0$ and $f_0 \neq 1$ is OK]

Interpretation: Toeplitz matrix $[f_{j-i}]_{i,j=1}^{\infty}$

Minors of this matrix $=$ coefficients of Schur expansion

Isaac Jacob Schoenberg

American mathematician

Isaac Jacob Schoenberg was a Romanian-American mathematician, known for his discovery of splines. Wikipedia

Born: April 21, 1903, Galați, Romania

Died: February 21, 1990, Madison, WI !

Education: Alexandru Ioan Cuza University

Academic advisor: Issai Schur

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Positive Schur expansions (characters)

$\Updownarrow$

Toeplitz matrices with positive minors (totally positive sequences)

T.p. sequences are used in applied math and numerical analysis, as pioneered by Schoenberg

Proof of Jacobi-Trudi formula:

Recall that $\sum_{n=0}^{\infty} h_n(y_1, y_2, \ldots)\, t^n = \prod_{i\geq 1} \frac{1}{1-ty_i}$

Hence, the Cauchy identity becomes

$$\sum_{\lambda \in Y} s_\lambda(x_1, x_2, \ldots)\, s_\lambda(y_1, y_2, \ldots) = \prod_{i\geq 1} \left(\sum_{n=0}^{\infty} h_n(y_1, y_2, \ldots)\, (x_i)^n \right)$$

$s_\lambda(y_1, y_2, \ldots)$ = coefficient c_λ in $\sum c_\lambda s_\lambda(x_1, \ldots)$ expansion of the right-hand side.

Therefore, $s_\lambda(y_1, y_2, \ldots) = \det\left[h_{\lambda_i - i + j}(y_1, y_2, \ldots) \right]$

$\blacksquare$

Examples:

$$s_{(n,0,0\dots)} = \det \begin{pmatrix} h_n & h_{n+1} & h_{n+2} & \cdots \\ 0 & 1 & h_1 & h_2 \cdots \\ & 0 & 1 & h_1 \cdots \\ & & & 1 \\ 0 & & & & \ddots \end{pmatrix} = h_n$$

$$s_{\underbrace{(1,1,\dots 1)}_{n}} = \det \left[h_{1-i+j} \right]_{i,j=1}^{n} = e_n$$

see Exercise in Lecture 3