Math 740 **Orthogonality** Lecture 5

Warm-up: Suppose that the number of variables $N = 1$.

Schur polynomials = monomials z^k, $k \in \mathbb{Z}$

Treat z as a complex number on the unit circle.

Lemma: z^k form an orthonormal basis with respect to the uniform measure on the unit circle:

complex conjugation

$\bar{z} = z^{-1}$ on circle

$$\frac{1}{2\pi}\int_0^{2\pi} \left(e^{i\varphi}\right)^k \overline{\left(e^{i\varphi}\right)}^{\,l} \, d\varphi = \frac{1}{2\pi i}\oint_{|z|=1} z^k \left(\bar{z}\right)^l \frac{dz}{z} = \delta_{k=l}$$

normalization constant

parameterization of the unit circle

contour integral

Jacobian of $z = e^{i\varphi}$

$$\begin{cases} 1, & k = l, \\ 0, & k \neq l. \end{cases}$$

Proof of Lemma:

A) $$\frac{1}{2\pi}\int_0^{2\pi} e^{i\varphi k}\cdot e^{-i\varphi \ell}\, d\varphi = \frac{1}{2\pi}\int_0^{2\pi}\left[\cos\varphi(k-\ell) + i\sin\varphi(k-\ell)\right] d\varphi$$

$$= \delta_{k=\ell} + i\cdot 0 = \delta_{k=\ell}$$

Important: essentially this is orthogonality of the Fourier basis $\{\cos\varphi k, \ \sin\varphi k\}_{k\in\mathbb{Z}}$

B) If you prefer the complex analysis language:

$$\frac{1}{2\pi i}\oint_{\substack{|z|=1 \\ \circlearrowleft}} z^k\cdot(\bar{z})^\ell \frac{dz}{z} = \frac{1}{2\pi i}\oint z^{k-\ell-1} = \operatorname*{Residue}_{\text{at } z=0}\left(z^{k-\ell-1}\right) = \delta_{k=\ell}$$

in this form contour can be any loop enclosing 0

Conclusion: At $N=1$, Schur polynomials form an orthogonal (Fourier) basis in $L_2(\mathbb{T})$, where $\mathbb{T}$ is unit circle $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$ and we use the uniform measure on $\mathbb{T}$.

Theorem: For each $N \geq 1$, $S_\lambda(z_1, \ldots z_N)$, $\lambda \in GT_N$ are orthonormal on N-dimensional torus $\mathbb{T}^N$ equipped with measure

$$\mu(d\theta_1, \ldots, d\theta_N) = \frac{1}{N!} \cdot \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^2 \cdot \frac{d\theta_1}{2\pi} \cdot \ldots \cdot \frac{d\theta_N}{2\pi}$$

angles parameterizing $\mathbb{T}^N$

For $\lambda \in GT_N$, $\mu \in GT_N$:

$$\frac{1}{N!(2\pi)^N} \int_0^{2\pi} \cdots \int_0^{2\pi} S_\lambda(e^{i\theta_1}, \ldots, e^{i\theta_N}) \overline{S_\mu(e^{i\theta_1}, \ldots e^{i\theta_N})} \cdot \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^2 \, d\theta_1 \cdot \ldots \cdot d\theta_n =$$

$$= \frac{1}{N!(2\pi i)^N} \cdot \oint \cdots \oint S_\lambda(z_1, \ldots z_N) \overline{S_\mu(z_1, \ldots z_N)} \cdot \prod_{a<b} |z_a - z_b|^2 \frac{dz_1}{z_1} \cdot \ldots \cdot \frac{dz_N}{z_N} = \delta_{\lambda=\mu}$$

Remarks: 1) green and blue forms are related by the change of coordinates $e^{i\theta_a} = z_a$

2) On the torus we have

$$S_\mu(z_1, \dots z_N) = S_\mu(z_1^{-1}, \dots, z_N^{-1}) = S_{-\mu}(z_1, \dots z_N)$$

$$|z_a - z_b|^2 = (z_a - z_b)(z_a^{-1} - z_b^{-1})$$

Using these, the blue integrand is converted into an analytic (holomorphic) expression with only singularities at 0 and ∞ in each z_a.

Then integration contour in each z_a can be chosen as an arbitrary positively oriented loop enclosing 0.

Proof of the orthogonality theorem:

$$S_\lambda(z_1,\dots z_N)\,\overline{S_\mu(z_1\dots z_N)}\,\prod|z_a-z_b|^2 =$$

$$S_\lambda(z_1,\dots z_N)\cdot\prod(z_a-z_b)\cdot S_\mu(z_1^{-1},\dots,z_N^{-1})\prod(z_a^{-1}-z_b^{-1}) =$$

$$=\mathrm{Alt}\left(z_1^{\lambda_1+N-1}\cdot\ldots\cdot z_N^{\lambda_N}\right)\cdot\mathrm{Alt}\left(z_1^{-(\mu_1+N-1)}\cdot\ldots\cdot z_N^{-\mu_N}\right)$$

alternating sums over permutations

We now have $(N!)^2$ monomials and we can integrate using the 1-dimensional orthogonality statement. The degrees of $z_1,\dots,z_N$ should match in order to get a non-zero result.

Hence: 1) If $\lambda\neq\mu$, then the integral vanishes

2) If $\lambda=\mu$, then $N!$ terms each integrate to $\frac{1}{N!}$ (integral had such prefactor!)

Notation: $\langle f, g\rangle_N$ — value of the scalar product, which makes Schur polynomials orthonormal, on a pair of functions f and g.

Question: On which space is $\langle\cdot,\cdot\rangle_N$ defined?

I: Algebraic point of view. If f and g are symmetric (perhaps, Laurent) polynomials, then

$$\langle f, g\rangle_N = \frac{1}{N!}\left[f(z_1,\dots z_N)\,g(z_1^{-1},\dots z_N^{-1})\prod_{a<b}(z_a-z_b)(z_a^{-1}-z_b^{-1})\right]_c$$

where $[Q]_c$ is the constant term of Q

Example: $[x_1^{-2}+x_2^{-2}+3+x_1+x_2]_c=3.$

Thus we do not need any actual integrations.

Leads to an axiomatic definition of Schur polynomials

Proposition: Suppose $\lambda_N \geq 0$, i.e. we deal with bona fide polynomials. Then $\{S_\lambda\}$, $\lambda \in GT_N \cap \mathbb{Y}$ are unique elements of Λ_N such that:

1) $S_\lambda = m_\lambda + (\text{lower degree terms})$
2) $\{S_\lambda\}$ are orthogonal with respect to $\langle \cdot, \cdot \rangle_N$

Remark: as definition of (lower degree terms) one can take an arbitrary linear extension of dominance order (making all elements of $\mathbb{Y}$ comparable)

For instance, lexicographic order (compare first coordinates if they are equal, compare second coordinates, etc)

Proof of proposition:

Applying Gram-Schmidt orthogonalization process to $\{m_\lambda\}$, $\lambda \in GT_N \cap Y$, we construct a unique system of symmetric polynomials, satisfying 1) and 2).

On the other hand, we already know that Schur polynomials $\{s_\lambda\}$ satisfy 1) and 2) $\blacksquare$

<u>Remark</u>: Such axiomatic definitions will be helpful in future lectures, when we define generalization of Schur polynomials, for which explicit detemінantal formulas do not exist

II: Analytic point of view. For any „good" function F on torus $T^N \subset \mathbb{C}^N$, symmetric in its N variables

we will not detail this in the class, a little bit of smoothness is enough as in 1d Fourier series.
F analytic/holomophic – more than enough

$$F(z_1, \dots, z_N) = \sum_{\lambda \in GT_N} c_\lambda \cdot s_\lambda(z_1, \dots, z_N),$$ where

converging on $|z_1| = \dots = |z_N| = 1$

$$c_\lambda = \langle F, s_\lambda \rangle_N = \frac{1}{N!(2\pi)^N} \int_0^{2\pi} \dots \int_0^{2\pi} F(e^{i\theta_1}, \dots, e^{i\theta_N}) \overline{s_\lambda(e^{i\theta_1}, \dots, e^{i\theta_N})} \cdot \prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^2 \, d\theta_1 \dots d\theta_N$$

Remark: When $N=1$ this is a decomposition of a function on circle (= function of angle $\theta \in [0, 2\pi)$) into Fourier series (sum of $e^{in\theta}$ or $\cos(n\theta)$, $\sin(n\theta)$)

Question: Schur polynomials are orthogonal on $\mathbb{T}^N$ with respect to a weird-looking measure. Why? Where did it come from?

The answer lies in representation-theoretic origin of Schur polynomials.

The material of the rest of Lec. 5 is supplementary. Feel free to skip.

$U(N)$ – group of all $N \times N$ complex unitary matrices.

A character of $U(N)$ is a continuous function

$$\chi : U(N) \to \mathbb{C},$$ such that

1) χ is central: $\chi(aba^{-1}) = \chi(b)$.

2) χ is positive-definite: $\forall k$, $\forall g_1, \dots, g_k \in U(N)$, $\forall z_1, \dots, z_k \in \mathbb{C}$

$$\sum_{a,b=1}^{k} z_a \overline{z_b}\, \chi(g_a g_b^{-1}) \geq 0$$ [non-negative definite matrix $\chi(g_a g_b^{-1})$]

How do you construct characters?

Take a finite-dimensional representation of $\mathcal{U}(N)$
$=$ homomorphism $T: \mathcal{U}(N) \to GL(V)$ with $\dim V < \infty$
invertible linear operators in V

General fact from representation theory of compact groups says that one can always choose a basis/scalar product in V, so that $T(u)$ are unitary, which implies $T(u^*) = T(u^{-1}) = (T(u))^* = (T(u))^{-1}$

[u^* = transpose and complex conjugate matrix to u
(you need to fix basis/scalar product to define $T(u)^*$)]

Proposition: $\chi(u) = \text{Trace}\,(T(u))$ is a character of $\mathcal{U}(N)$

Proof: 1) $\chi(vuv^{-1}) = \text{Trace}(T(v)\,T(u)\,T(v)^{-1}) = \text{Trace}\,(T(u)) = \chi(u)$

2) $\sum z_i \bar{z}_j \chi(g_i g_j^{-1}) = \text{Trace}\,(AA^*)$ $\quad A = \sum z_i T(g_i)$

$=$ sum of (squared) singular values of A ≥ 0 $\blacksquare$

Theorem: Set of characters of $\mathcal{U}(N)$ forms a (positive) cone spanned by characters of irreducible representations.

$\mathcal{U}(N)$ is not important here, it holds for any compact group $\approx$ any representation decomposes into irreducibles

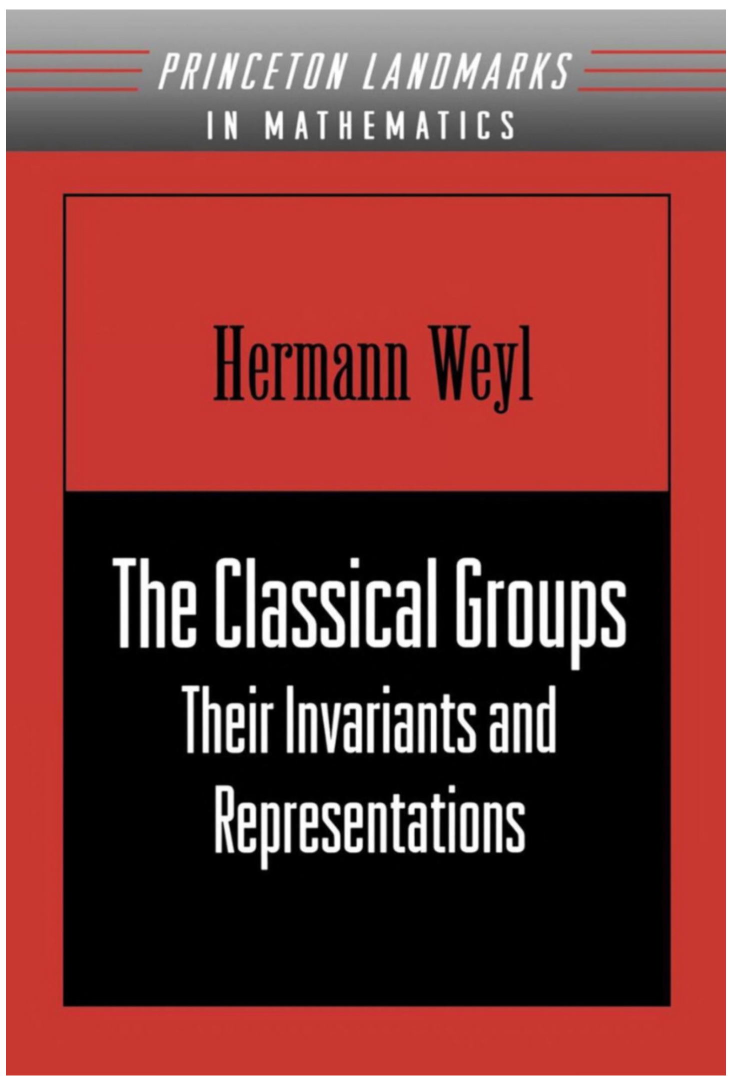
We are not giving a proof in this class.

Irreducible representation $T: \mathcal{U}(N) \to GL(V)$ is such that the only (linear) subspaces $W \subset V$ invariant under T ($T(u)\, W \subset W$ for each $u \in \mathcal{U}(N)$) are $\{0\}$ and V.

Being a cone means that each character $\chi(u)$ has a unique decomposition

$$\chi(u) = \sum_{\lambda} c_\lambda \chi^\lambda(u) \quad \Big| \quad c_\lambda \geq 0, \ \chi^\lambda \text{ – character of irred. representation } \lambda$$

Character of $\mathcal{U}(N)$ is a conjugation-invariant function of a matrix $\Rightarrow$ it is a symmetric function of eigenvalues of this matrix, which are complex numbers on the unit circle: $\chi(u_1, \ldots u_N)$, $\{u_i\}$ are e.v. of $U \in \mathcal{U}(N)$.



Theorem: (no proof in this class)

Irreducible representations of $\mathcal{U}(N)$ are parameterized by $\lambda \in GT_N$.

The characters are Schur polynomials

$$\chi^{\lambda}(U) = S_{\lambda}(u_1, \ldots u_N), \qquad U \in \mathcal{U}(N)$$

$u_1, \ldots, u_N \in \mathbb{T}$ are eigenvalues of U

In particular, $S_\lambda(1, \ldots 1) = \mathrm{Trace}(\mathrm{Id}) =$ Dimension of the representation space

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Hence, all characters of $\mathcal{U}(N)$ are combinations of Schur functions: $\chi(u_1, \ldots u_N) = \sum_{\lambda} c_\lambda S_\lambda(u_1, \ldots u_N)$, $c_\lambda \geq 0$.

Peter-Weyl theorem (no proof in this class) implies that for any compact group characters of irreducible representations form an orthonormal basis in L_2 (conjugation invariant functions on group) with respect to Haar ($\approx$ uniform) measure.

$\mathcal{U}(N) \subset \mathbb{C}^{N^2} = \mathbb{R}^{2N^2}$ is a smoothly embedded submanifold we can restrict Lebesgue measure (usual area, volume, etc) onto $\mathcal{U}(N)$ and renormalize, so that measure$(\mathcal{U}(N)) = 1$.
This is the Haar measure on $\mathcal{U}(N)$.

Conclusion: Schur polynomials $s_\lambda(u_1, \dots u_N)$ should be orthonormal with respect to the uniform measure, if we treat them as functions on $\mathcal{U}(N)$.

Theorem: (one of the first computations of the random matrix theory; no proof given in this class)

For $u \in \mathcal{U}(N)$ let $\rho(u)$ be N-tuple $0 \le \theta_1 \le \dots \le \theta_N < 2\pi$ of eigenvalues of u. Then the ρ-pushforward of the Haar (uniform) measure on $\mathcal{U}(N)$ is

$$\prod_{a<b} |e^{i\theta_a} - e^{i\theta_b}|^2 \, d\theta_1 \cdot \ldots \cdot d\theta_N$$

This is precisely the measure from the first part of lecture up to $\frac{1}{N!}$ prefactor coming from fixed order $\theta_1 < \theta_2 < \dots < \theta_N$

Conclusion: Orthogonality of Schur polynomials on the torus $\mathbb{T}^N$ is ρ-projection of their orthogonality as characters of irred. representations of $\mathcal{U}(N)$.