Math 740 Ring of symmetric functions Lecture 2

Def. $\Lambda_N = \left\{ \begin{array}{l} \text{polynomials in } x_1, x_2, \dots, x_N, \text{ invariant} \\ \text{under permutations of indices } 1, \dots, N \end{array} \right\}$

Coefficients can be from $\mathbb{Z}$, $\mathbb{Q}$, $\underbrace{\mathbb{R}}$, or $\mathbb{C}$

can stick to this for simplicity

Λ_N is a graded ring, meaning that

- You can add and multiply its elements (with all usual axioms)
- $\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k$, where Λ_N^k consists of all degree k homogeneous polynomials (and 0)

$f \in \Lambda_N^k$, $g \in \Lambda_N^m \Rightarrow f \cdot g \in \Lambda_N^{k+m}$

$$\Lambda_N = \bigoplus_{k \geq 0} \Lambda_N^k$$

$\Lambda_N^0 = \text{constants} = \langle 1 \rangle$

$\Lambda_N^1 = \langle x_1 + \ldots + x_N \rangle$

$\Lambda_N^2 = \langle x_1^2 + x_2^2 + \ldots + x_N^2, \sum_{1 \leq i < j \leq N} x_i x_j \rangle$

What is a linear basis of Λ_N^k?

Definition: take $\lambda = (\lambda_1 \geq \ldots \geq \lambda_N \geq 0)$ integers, $\sum \lambda_i = k$

Monomial symmetric polynomial $m_\lambda \in \Lambda_N^k$

$$m_\lambda = \sum_{\substack{(\mu_1, \ldots \mu_N) \\ \text{permutations} \\ \text{of } (\lambda_1, \ldots \lambda_N)}} x_1^{\mu_1} \cdot \ldots \cdot x_N^{\mu_N}$$

$N=3$ examples

$m_{(1,0,0)} = x_1 + x_2 + x_3$

$m_{(2,0,0)} = x_1^2 + x_2^2 + x_3^2$

$m_{(1,1,0)} = x_1 x_2 + x_1 x_3 + x_2 x_3$

$m_{(1,1,1)} = x_1 x_2 x_3$

Notations for labels of symmetric polynomials

Signature λ of rank N

can be negative

N-tuple of integers $(\lambda_1 \geq \dots \geq \lambda_N)$

set of all such signatures is $\mathbb{GT}_N$

Gelfand and Tsetlin

Partition λ of n

parts of λ

sequence of integers $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$, such that $\sum_{i \geq 1} \lambda_i = n$

set of all such partitions is $\mathbb{Y}_n$ Young

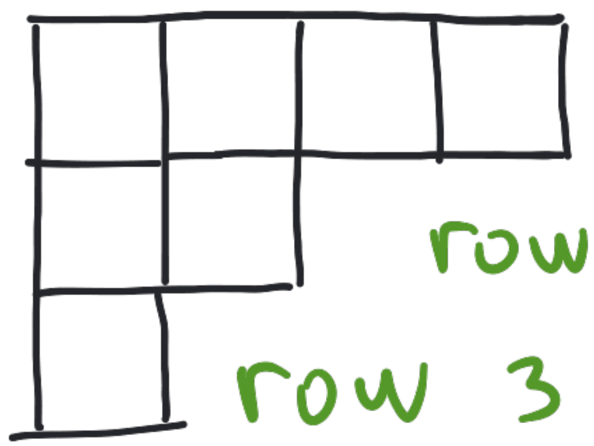
<u>Proposition</u>: $\Lambda_N^K = \langle m_\lambda \rangle_{\lambda \in \mathbb{GT}_N \cap \mathbb{Y}_K}$:

real coefficients

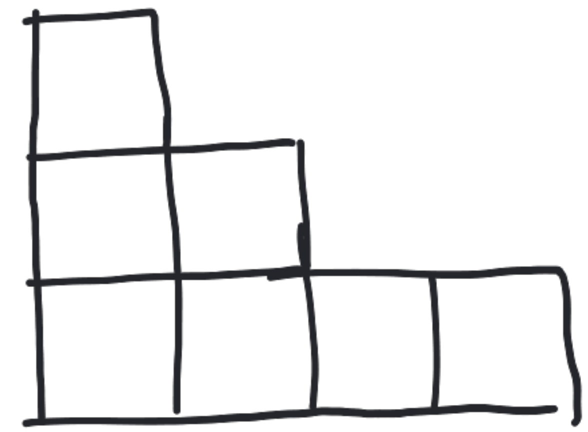
each $f \in \Lambda_N^K$ has a (unique) representation $f = \sum_{\lambda \in \mathbb{GT}_N \cap \mathbb{Y}_K} c_\lambda \cdot m_\lambda$

Partitions = Young diagrams

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant 0 \qquad \sum_i \lambda_i = n$$

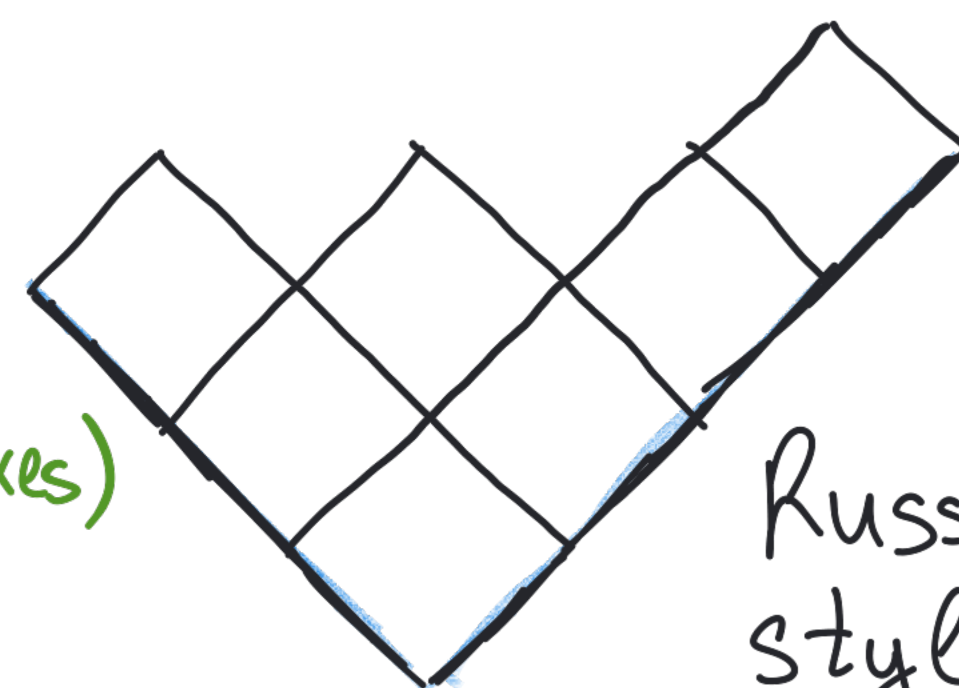


$n = 7$



French style

English style



Russian style

$|\lambda| = \sum \lambda_i = n$ (number of boxes)

$\ell(\lambda)$ = length of the 1st column

More partitions notations:

Given $\lambda_1 \geq \lambda_2 \geq \dots - \geq 0 \geq 0 \geq 0 \dots$,

$$\ell(\lambda) = \max\{ i \mid \lambda_i > 0 \}$$

We often *identify* a partition λ with $\tilde{\lambda} \in GT_M$ for any $M \geq \ell(\lambda)$.

Example: $(5,2,1)$ is a partition of 8 $(\in \mathbb{Y}_8)$ and $\in GT_3$, and $\in GT_4$ as $(5,2,1,0)$, and $\in GT_5$ as $(5,2,1,0,0)$, and etc

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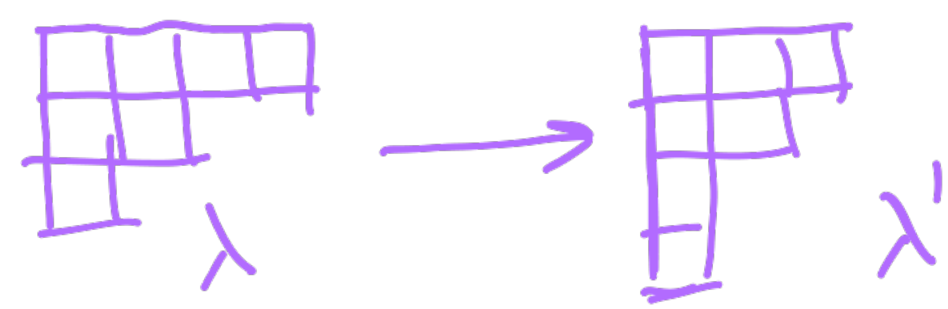
Transposition $\lambda \to \lambda'$

$$\lambda'_i = \{ j \geq 1 : \lambda_j \geq i \}$$



rows $\longrightarrow$ columns

Back to Λ_N – symmetric polynomials in $(x_1, \dots, x_N)$

Question: what is an algebraic basis of Λ_N?

[In linear basis we can only add/multiply by constants.
In algebraic basis we are allowed to multiply]

Fundamental theorem of symmetric polynomials:

As rings/algebras $\Lambda_N \cong \mathbb{R}[e_1, e_2, \dots, e_N]$

$$e_k = \sum_{1 \le i_1 < \dots < i_k \le n} x_{i_1} \cdot x_{i_2} \cdot \dots \cdot x_{i_k}$$ | elementary symmetric polynomials

In other words, $\forall f \in \Lambda_N$, there exist a unique polynomial g in N-variables, such that

$$f = g(e_1, e_2, \dots e_N)$$

What are $e_1, e_2, \dots, e_N$? Vieta's formulas:

$$(z+x_1)(z+x_2)\cdot\ldots\cdot(z+x_N) = z^N + z^{N-1}e_1(x_1,\dots x_N) +$$
$$+ z^{N-2}e_2(x_1,\dots x_N) + \dots + e_N(x_1, \dots x_N)$$

Proof of Fundamental theorem:

Step 1. Recall $\Lambda_N = \langle m_\lambda \rangle\big|_{\lambda \in GT_N,\ \lambda_N \geq 0}$ (linear basis)

We introduce a partial order on partitions

$\lambda \geq \mu$ if $|\lambda| = |\mu|$ and $\forall i \geq 1$ $\quad \lambda_1 + \lambda_2 + \dots + \lambda_i \geq \mu_1 + \mu_2 + \dots + \mu_i$

dominance order

[many partitions are incomparable!]

Graphically

$\lambda \geq \mu$

means that μ is obtained from λ by moving boxes down.

Exercise: $\lambda \geq \mu$ if and only if $\lambda' \leq \mu'$

Step 2. For a partition λ set $e_\lambda = \prod_{i \geq 1} e_{\lambda_i}$

Proposition:
$$e_{\lambda'} = m_\lambda + \sum_{\substack{\mu \mid \mu \leq \lambda \\ \mu \neq \lambda}} c_\lambda^\mu \cdot m_\mu$$
coefficients

Proof: Each monomial in $e_{\lambda'}$ has the form

$$(x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_{\lambda_1'}})(x_{j_1} \cdot x_{j_2} \cdot \ldots \cdot x_{j_{\lambda_2'}}) \cdot \ldots$$

$i_1 < i_2 < \ldots < i_{\lambda_1'}$ $\qquad$ $j_1 < j_2 < \ldots < j_{\lambda_2'}$

Expand $e_{\lambda'} = \sum_\mu c_\lambda^\mu \cdot m_\mu$. Our aim: show $\mu \leq \lambda$ whenever $c_\mu \neq 0$.

Hence, $\deg(x_1)$ in each monomial is $\leq$ # of factors

# of factors $= \lambda_1 \implies \mu_1 \leq \lambda_1$

Furter, $\deg(x_1)+\deg(x_2)$ in each monomial is $\leq$ 2 # of factors with at least 2 variables + # of factors with precisely 1 variable $= \lambda_1+\lambda_2$

Hence, $\mu_1+\mu_2 \leq \lambda_1+\lambda_2$.

Continuing in the same way we conclude that $\mu \leq \lambda$, whenever $c_\lambda^\mu \neq 0$.

Remains to show that $c_\lambda^\lambda = 1$.

But the only way to get monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdot x_3^{\lambda_3} \cdots x_n^{\lambda_n}$ is by multiplying $(x_1 x_2 \cdot \ldots \cdot x_{\lambda_1'}) \cdot (x_1 x_2 \cdot \ldots \cdot x_{\lambda_2'}) \cdot \ldots$

Step 3 (of the proof of fundamental theorem)

We can now show by induction that each m_λ can be represented as

$$m_\lambda = g(e_1, e_2, \dots, e_N)$$

Indeed, $m_\lambda - e_{\lambda'} = \sum_{\mu < \lambda} c_\lambda^\mu m_\mu$ and we can use the induction assumption.

(The base of induction is $m_{(1,1,\dots 1)} = e_N$)

Step 4

Remains to show the uniqueness.

We argue by contradiction.

If $g = f_1(e_1 \dots e_N) = f_2(e_1, \dots, e_N)$, then
$(f_1 - f_2)(e_1, \dots e_N) = 0$

Write

$$0=(f_1-f_2)(e_1,\dots e_N)=\sum_\lambda c_\lambda e_\lambda \qquad \Big| \text{ finite sum}$$

In this sum choose λ with maximal $|\lambda|$

If there are several such λ's, choose among them one which is minimal in dominance order

(since the order is partial, there might be several)

Then λ' is maximal $\Rightarrow$ in decomposition of $(f_1-f_2)(e_1,\dots e_N)$ into m_μ, $m_{\lambda'}$ appears with non-zero coefficient c_λ (by Proposition)

This contradiction finishes the proof of theorem.

$e_1, e_2, \dots \quad e_N \in \Lambda_N$ elementary symmetric functions

$p_1, p_2, p_3, \dots \in \Lambda_N$ power sums

$$p_k = \sum_{i=1}^{N} (x_i)^k$$

$h_1, h_2, h_3, \dots \in \Lambda_N$ complete homogeneous symmetric polynomials

$$h_k = \sum_{1 \le i_1 \le \dots \le i_k \le N} x_{i_1} \cdot \dots \cdot x_{i_k}$$

Since $\Lambda_N = \mathbb{R}[e_1, \dots e_N]$, $\{p_k\}$ and $\{h_k\}$ must be algebraically dependent!

Example: $N=2 \,|\; p_1^3 - p_3 - 3p_1\left(\frac{p_1^2 - p_2}{2}\right) = 0$

Indeed, $(x_1+x_2)^3 = x_1^3 + x_2^3 + 3(x_1+x_2)x_1x_2 = x_1^3 + x_2^3 + 3(x_1+x_2)\left(\frac{(x_1+x_2)^2 - x_1^2 - x_2^2}{2}\right)$

Observation: In all the arguments so far the number of variables (N) was not important.

This is often the case: we do not care about N as long as it is large enough.

Next task: get rid of N by sending $N \to \infty$.

Definition: For $N < M$ let $\rho^k_{M\to N} : \Lambda^k_M \to \Lambda^k_N$ be the map setting $x_{N+1} = x_{N+2} = \dots = x_M = 0$.

We have
$$\rho^k_{M\to N}(m_\lambda) = \begin{cases} m_\lambda, & \text{if } \ell(\lambda) \leq N, \\ 0, & \text{if } \ell(\lambda) > N. \end{cases}$$

Definition: Λ^k = projective limit of Λ^k_N as $N \to \infty$.

Elements of Λ^k are sequences of polynomials

homogeneous of degree k

$$f_1 \xleftarrow{p^k_{2\to1}} f_2 \xleftarrow{p^k_{3\to2}} f_3 \xleftarrow{p^k_{4\to3}} \cdots$$

in 1 variable

in 2 variables

in 3 variables

Λ^k is a linear space with basis $\{m_\lambda\}_{\lambda\in Y_k}$

Definition: The ring of symmetric functions

$$\Lambda = \bigoplus_{k\geq 0} \Lambda^k$$

Examples: $1 \in \Lambda$. $x_1+x_2+x_3+\ldots \in \Lambda$

$\sum_{i<j} x_i x_j \in \Lambda$

Question: why did not we simply define $p_{M\to N}: \Lambda_M \to \Lambda_N$ and consider projective limit of Λ_N?

Answer: we want to preserve grading!

$$\sum_i x_i + \sum_{i<j} x_i x_j + \sum_{i<j<k} x_i x_j x_k + \sum_{i<j<k<l} x_i x_j x_k x_l + \ldots \in \lim_{N\to\infty} \mathrm{proj}\, \Lambda_N$$

(because by setting variables $=0$ we get a sequence)

But this is **not** an element of Λ.

Theorem: As a vector space $\Lambda = \langle m_\lambda \rangle \big|_{\lambda \in Y}$ finite linear combinations $Y = \bigcup_{k\geq 0} Y_k$

As an algebra $\Lambda = \mathbb{R}[e_1, e_2, e_3, \ldots]$
finite degree polynomials in these variables.

Proof: the same as for Λ_N. $\blacksquare$