Math 740 Lecture 3

Λ_N - symmetric polynomials in $(x_1, \ldots x_N)$

Λ - symmetric functions in $(x_1, x_2, \ldots)$

= symmetric power series of bounded degree

Sometimes we also deal with series in Λ

$\Lambda[[t]]$ - all series of the form $\underset{\text{in } \Lambda}{f_0} + t \underset{\text{in } \Lambda}{f_1} + t^2 \underset{\text{in } \Lambda}{f_2} + \ldots$

$\hat{\Lambda}$ - all series of the form $\underset{\text{in } \Lambda^0}{g_0} + \underset{\text{in } \Lambda^1}{g_1} + \underset{\substack{\text{(degree 2)} \\ \text{in } \Lambda^2}}{g_2} + \ldots$

These all are algebras:

One can add and multiply their elements

Three families of algebraic generators of Λ

1. $e_k = \sum_{i_1 < \ldots < i_k} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k} = m_{(1^k)}$

elementary symmetric functions

$e_\lambda = \prod_{i \geq 1} e_{\lambda_i}$ each $f \in \Lambda$ has a unique decomposition

$f = \sum_\lambda c_\lambda \cdot e_\lambda$ (see last lecture)

Their generating function

$$E(t) = \sum_{k=0}^{\infty} e_k t^k = 1 + e_1 t + e_2 t^2 + \ldots$$

Lemma: $E(t) = \prod_{i \geq 1} (1 + t x_i)$

Proof: Open the paranthesis and compute the coefficient of t^k matching the definition of e_k $\boxtimes$

2. $h_k = \sum_{i_1 \le i_2 \le \ldots \le i_k} x_{i_1} \cdot x_{i_2} \cdot \ldots \cdot x_{i_k} = \sum_{\lambda : |\lambda| = k} m_\lambda$

complete homogeneous symmetric functions

Their generating function

$$H(t) = \sum_{k \geq 0} h_k t^k = 1 + h_1 t + h_2 t^2 + \ldots$$

Lemma: $H(t) = \prod_{i \geq 1} (1 - t x_i)^{-1}$

Proof: $\prod (1 - t x_i)^{-1} = \prod \left(1 + t x_i + t^2 (x_i)^2 + t^3 (x_i)^3 + \ldots\right)$

Open the paranthesis and compute the coefficient of t^k matching the definition of h_k ■

$$E(t) = \prod (1 + t x_i) \qquad H(t) = \prod (1 - t x_i)^{-1}$$

Corollary: $E(t)H(-t) = 1$, which means

For each $n > 0$ $\quad \sum_{k=0}^{n} h_k e_{n-k} (-1)^k = 0 \qquad (*)$

$$e_n - h_1 e_{n-1} + h_2 e_{n-2} + \dots + (-1)^n h_n = 0$$

[with notation $e_0 = h_0 = 1$
$e_{-k} = h_{-k} = 0$, $k < 0$]

Exercise: $(*)$ implies that

$$e_n = \det \left(h_{1-i+j} \right)_{i,j=1}^{n} \qquad h_n = \det \left(e_{1-i+j} \right)_{i,j=1}^{n}$$

Definition: $\omega : \Lambda \to \Lambda$ is an algebra homomorphism such that $\omega(e_n) = h_n$

Theorem: $\omega^2 = \mathrm{Id}$ and, therefore, ω is an automorphism

Proof: $\omega = \omega^{-1}$ due to symmetry of $(*)$ under $e_i \leftrightarrow h_i$. ∎

Corollary: h_k are algebraic generators of Λ

$h_\lambda = \prod_{i \geq 1} h_{\lambda_i}$. Each $f \in \Lambda$ has a unique decomposition

$$f = \sum_\lambda c_\lambda \cdot h_\lambda$$

Proof] $w(f) = \sum_\lambda c_\lambda \cdot e_\lambda$, apply $w(\cdot)$ to both sides $\blacksquare$

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3. $p_k = \sum_{i \geq 1} (x_i)^k$ (Newton) power sums

Their generating function

$$P(t) = \sum_{k \geq 1} \frac{p_k t^k}{k} = p_1 t + \frac{p_2}{2} t^2 + \frac{p_3}{3} t^3 + \dots$$

Lemma: $\exp(-P(t)) = \prod_{i \geq 1} (1 - x_i t)$

Proof of Lemma:

$$\exp(-P(t)) = \exp\Big(\sum_{i\geq 1}\Big(-x_i t - \frac{x_i^2 t^2}{2} - \frac{x_i^3 t^3}{3} - \dots\Big)\Big)$$

$$\ln(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \dots$$

$$= \exp\Big(\sum_{i\geq 1} \ln(1-x_i t)\Big) = \prod_{i\geq 1}(1-x_i t). \quad \blacksquare$$

Corollary:

$$\exp(P(t)) = H(t) \qquad (\mathrm{I})$$

$$\exp(-P(-t)) = E(t) \qquad (\mathrm{II})$$

Proof: $(\mathrm{I}) = \prod(1-x_i t)^{-1}$; $(\mathrm{II}) = \prod(1+x_i t)$. $\blacksquare$

Theorem (Newton identities)

For each $k\geq 1$

$$k e_k = \sum_{i=1}^{k} (-1)^{i-1} p_i e_{k-i}$$

$$k h_k = \sum_{i=1}^{k} p_i h_{k-i}$$

Proof of Newton identities:

Differentiate (II) to get

$$P'(-t)\exp(-P(-t)) = E'(t) = \sum_{k=1}^{\infty} k e_k t^{k-1}$$

$\parallel$ use (II) again

$$P'(-t)\,E(t)$$

$\parallel$

$$\left(\sum_{k\geq 1} p_k t^{k-1}(-1)^{k-1}\right)\left(\sum_{k\geq 0} t^k e_k\right)$$

Comparing the coefficient of t^{k-1} in both sides we prove the first Newton identity.

The second one is proven in the same way $\blacksquare$

Corollary of Newton identities:

$p_1, p_2, \dots$ are algebraic generators of Λ

$p_\lambda = \prod_{i \geq 1} p_{\lambda_i}$, each $f \in \Lambda$ has a unique decomposition

$$f = \sum_\lambda c_\lambda \cdot p_\lambda$$

Proof: We can rewrite the second identity as

$$h_k = \frac{p_k}{k} + \text{polynomial of } (p_1, \dots p_{k-1})$$

Hence, plugging into $f = \sum c_\lambda \cdot h_\lambda$, we get the desired decomposition.

Other way round, we also have $p_k = k h_k + \text{polynomial of } (h_1, \dots h_{k-1})$

Hence, if p_k were algebraically dependent, so would be h_k. Therefore, decomposition is unique. $\blacksquare$

$$\Lambda = \mathbb{R}[e_1, e_2, \ldots] = \mathbb{R}[h_1, h_2, \ldots] = \mathbb{R}[p_1, p_2, \ldots]$$

$$= \langle m_\lambda \rangle_{\lambda \in Y}$$

Remark 1: Project $\Lambda \to \Lambda_N$ by $\rho_{\infty, N}$ (setting $x_{N+1} = x_{N+2} = \ldots = 0$)

- $e_{N+k} \to 0$, $k > 0$
- $m_\lambda \to 0$, if $\ell(\lambda) > N$
- $\{h_k\}$ and $\{p_k\}$ become dependent

Remark 2: $m_\lambda \in \mathbb{Z}[e_1, e_2, \ldots]$, $m_\lambda \in \mathbb{Z}[h_1, h_2, \ldots]$

But $m_\lambda \notin \mathbb{Z}[p_1, p_2 \ldots]$ in general

Example: $\sum_{i<j} x_i x_j = e_2 = h_1^2 - h_2 = \frac{1}{2}(p_1^2 - p_2)$

integer coefficients need to divide

Recall the involution $\omega : \Lambda \to \Lambda$

$\omega(e_k) = h_k$, $\omega(h_k) = e_k$

What about p_k?

Proposition: $\omega(p_k) = (-1)^{k-1} p_k$

Proof: Act with ω on the first Newton identity and compare the result with the second Newton identity $\blacksquare$

Application of Newton identities by

(perhaps, the oldest use of theory of symmetric functions)

Leonhard Euler

Portrait by Jakob Emanuel Handmann (1753)

Born	15 April 1707 Basel, Switzerland
Died	18 September 1783 (aged 76) [OS: 7 September 1783] Saint Petersburg, Russian Empire

Task: Compute $\zeta(2k) = \sum_{n \geq 1} \frac{1}{n^{2k}}$

Starting point: $(*)$ $\frac{\sin(\pi x)}{\pi x} = \prod_{n \geq 1} \left(1 - \frac{x^2}{n^2}\right)$

can be proven through, e.g.:

- complex analysis — comparing zeros
- Fourier analysis — expanding $\cos(\alpha x)$ on $[-\pi, \pi]$ in the sum of $\cos(nx)$ and $\sin(nx)$

We take $(*)$ for granted

Expand $\sin(\pi x)$ in series and set $t=-x^2$, converting (*) into:

$$\sum_{k=0}^{\infty} \frac{(\pi^2 t)^k}{(2k+1)!} = \prod_{n\geq 1}\left(1+t\cdot\frac{1}{n^2}\right)$$

$$\parallel$$

$$1+\frac{\pi^2}{6}t+\frac{\pi^4}{120}t^2+\frac{\pi^6}{5040}t^3+\ldots$$

Take variables $x_i = \frac{1}{i^2}$, $i=1,2,\ldots$

Then $RHS = E(t) = \sum_{k\geq 0} e_k t^k \Rightarrow e_k = \frac{\pi^{2k}}{(2k+1)!}$

Task: compute p_k, $k=1,2,\ldots$

- $\sum_{n=1}^{\infty}\frac{1}{n^2} = p_1 = e_1 = \frac{\pi^2}{6}$

Newton identity

- $\sum_{n=1}^{\infty}\frac{1}{n^4} = p_2 = -2e_2 + p_1 e_1 = -2e_2 + (e_1)^2 = -2\frac{\pi^4}{120}+\left(\frac{\pi^2}{6}\right)^2 = \frac{\pi^4}{90}$

- $$\sum_{n=1}^{\infty} \frac{1}{n^6} = p_3 \overset{\text{Newton identity}}{=} 3e_3 - e_2 p_1 + p_2 e_1 =$$

$$= 3 \cdot \frac{\pi^6}{5040} - \frac{\pi^4}{120} \cdot \frac{\pi^2}{6} + \frac{\pi^4}{90} \cdot \frac{\pi^2}{6} = \frac{\pi^6}{945}$$

- $$\sum_{n=1}^{\infty} \frac{1}{n^8} = \text{----} \qquad \text{Exercise: compute this!}$$