Math 740 (Symmetric functions) Lecture 1

What are symmetric functions?
Why should we care about them?

Basic definition: we deal with polynomials in $x_1, x_2, x_3, \ldots$, which are symmetric under permutations of variables

Example 1: $x_1^2+x_2^2+x_3^2$ is a symmetric polynomial in 3 variables (x_1, x_2, x_3)
[but not in 4 variables!]

Example 2: $x_1^2 x_2 + x_3$ is not symmetric

One can add and multiply them!

Central question: given a („complete") family of symmetric polynomials $\{P_\lambda\}$, how do you expand an arbitrary Q?

$$Q = \sum_\lambda c_\lambda P_\lambda$$

how to find these?

Example: Take $N=3$ variables,

$e_0 = 1$, $e_1 = x_1 + x_2 + x_3$, $e_2 = x_1x_2 + x_1x_3 + x_2x_3$, $e_3 = x_1x_2x_3$

and all products of these 4 polynomials.

How do you expand $p_3 = (x_1)^3 + (x_2)^3 + (x_3)^3$?

Answer: $p_3 = e_1^3 - 3e_1e_2 + 3e_3$

Check: $(x_1+x_2+x_3)^3 - 3(x_1+x_2+x_3)(x_1x_2+x_2x_3+x_1x_3)$
$+3x_1x_2x_3 = ?$

Difficulty: many variables, large degrees complicated families of „natural" polynomials.

This class: introduce and study such families and ways to transition between them.

Motivation: this is a language for proving many deep theorems in other areas of mathematics

Motivation 1: combinatorics

Example: $A \times B \times C$ boxed plane partition

$C=7$ $\quad A=3$

$B=3$

7	5	2
3	2	0
1	1	0

$A \times B$ table filled with ≥ 0, $\leq C$ integers weakly decreasing along rows/columns

Theorem (Macmahon 1896) $\#\, A \times B \times C$ bpp is

$$\prod_{a=1}^{A}\prod_{b=1}^{B}\prod_{c=1}^{C} \frac{a+b+c-1}{a+b+c-2}$$

$\boxed{\text{Exercise: this formula gives an integer}}$

Modern proof: compute via

Schur symmetric polynomial $\rightarrow$

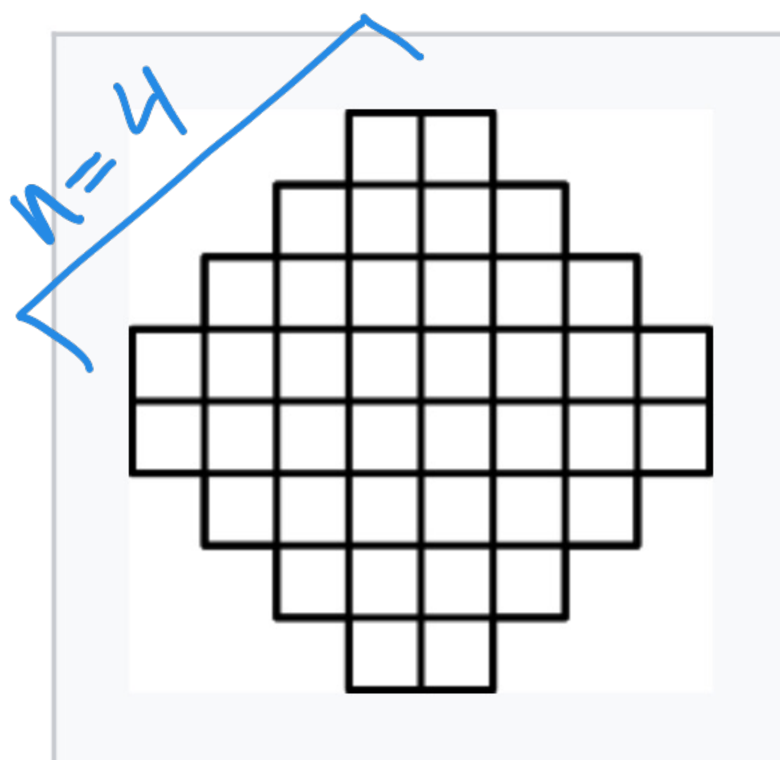
$$s_{(C^A,0^B)}(x_1,\dots x_{A+B})\Big|_{x_1=\dots=x_{A+B}=1}$$

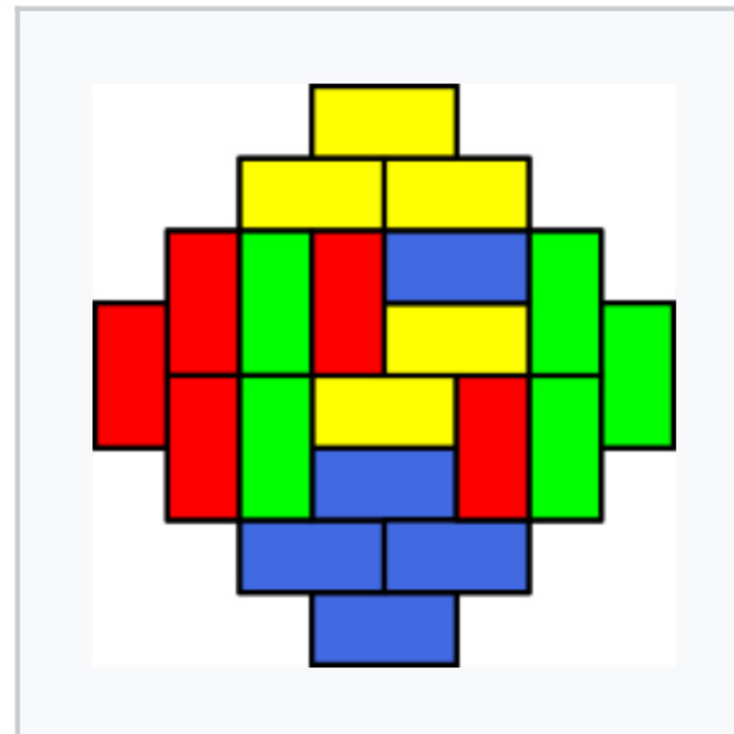
Explicit evaluation of specialization

Motivation 1: combinatorics

Example: count all domino tilings of $n \times n$ Aztec diamond

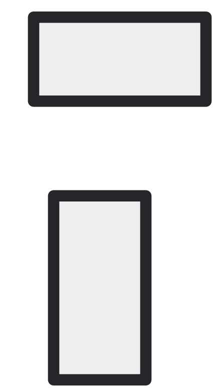
(from wikipedia)





One possible tiling (ignore colors)

2x1 dominos



Theorem: $\#\text{tilings} = 2^{n(n+1)/2}$

Published: September 1992

Alternating-Sign Matrices and Domino Tilings (Part I)

Noam Elkies, Greg Kuperberg, Michael Larsen & James Propp

Journal of Algebraic Combinatorics **1**, 111–132(1992) | Cite this article

Proof idea: in fact, it is

$$\prod_{k=1}^{n} \underbrace{\prod_{i=1}^{k} (1+x_i)}_{\text{symmetric polynomials}} \Big|_{x_1 = \ldots = x_n = 1}$$

Motivation 2: representation theory

Example:

homomorphism

$$T : \underbrace{\mathcal{U}(N)}_{\text{group of } N\times N \text{ unitary matrices}} \longrightarrow GL(V)$$

representation

invertible operators

finite-dimensional vector space

Fact 1: each representation = direct sum of irreducibles

Fact 2: irreducibles are parameterized by $\lambda_1 \geq \ldots \geq \lambda_N$ (integers)

tensor product, direct sum

$$T_\lambda \otimes T_\mu = \bigoplus_\nu C^\nu_{\lambda\mu} T_\nu$$

irreducible representations

multiplicities = Littlewood-Richardson coefficients

Fact 3:

$$S_\lambda(x_1, \ldots x_N) \cdot S_\mu(x_1, \ldots x_N) = \sum_\nu C^\nu_{\lambda\mu} S_\nu(x_1, \ldots x_N)$$

Schur symmetric polynomials

Motivation 3: classical analysis (integral computations)

Example: Selberg, Atle Remarks on a multiple integral. (Norwegian) *Norsk Mat. Tidsskr.* 26 (1944), 71–78.

$$S_n(\alpha,\beta,\gamma) = \int_0^1 \cdots \int_0^1 \prod_{i=1}^n t_i^{\alpha-1}(1-t_i)^{\beta-1} \prod_{1\le i<j\le n} |t_i - t_j|^{2\gamma}\, dt_1 \cdots dt_n$$

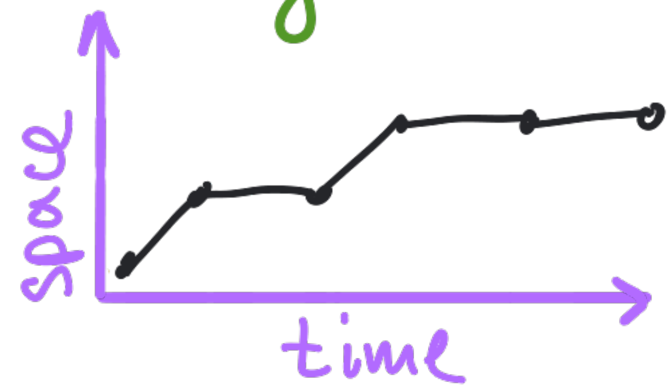
$$= \prod_{j=0}^{n-1} \frac{\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)\Gamma(1+(j+1)\gamma)}{\Gamma(\alpha+\beta+(n+j-1)\gamma)\Gamma(1+\gamma)}$$

A possible modern proof:

This is a continuous limit of Cauchy (-Littlewood) summation identity for (symmetric) Macdonald polynomials

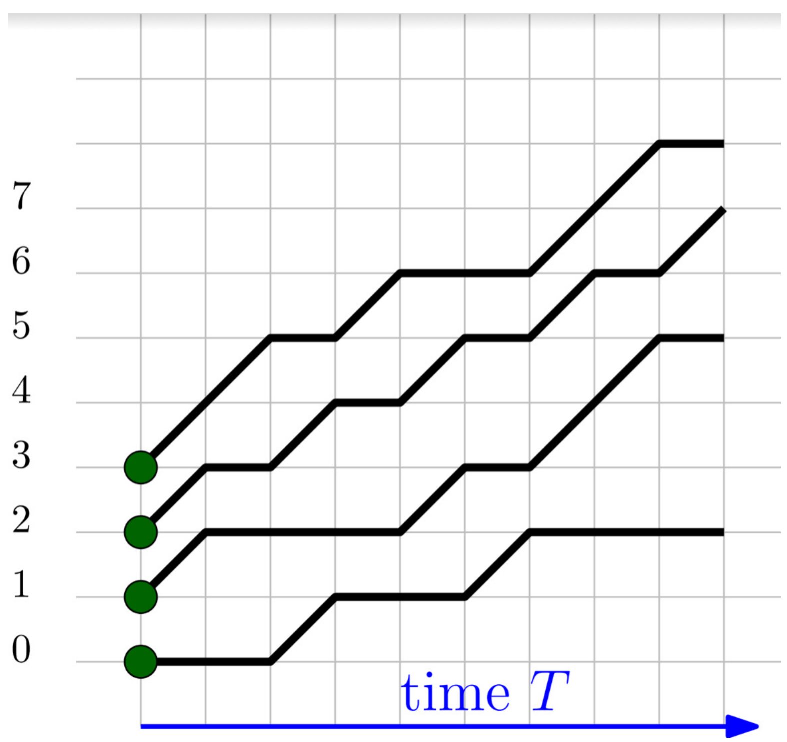
Motivation 4: probability

Example:



Random walk on $\mathbb{Z}$

Consider N independent random walks



- started at $0, 1, \dots N-1$
- conditioned to never collide

How to analyze as $N, T \to \infty$?

Theorem: $$\mathbb{E}\left(\sum_{i=1}^{N} x_i^{\kappa}(T)\right)^m = (\mathcal{D}_\kappa)^m \underbrace{\prod_{i=1}^{N}(1-p+px_i)^T}_{\text{symmetric polynomial}}\Bigg|_{x_1=\dots=x_N=1}$$

symmetric differential operator

$$\mathcal{D}_\kappa = \prod_{i<j}(x_i - x_j)^{-1} \sum_{i=1}^{N}\left(x_i \frac{\partial}{\partial x_i}\right)^{\kappa} \prod_{i<j}(x_i - x_j)$$

Motivation 4: probability

Example: Take a uniformly random permutation of n letters $\{1,2,\dots,n\}$ $\quad 7\,\underset{\sim}{1}\,\underset{\sim}{2}\,\underset{\sim}{4}\,\underset{\sim}{5}\,3\,\underset{\sim}{6}\,\underset{\sim}{8}$

ℓ_n = length of the longest increasing subsequence
($\ell_n = 6$ above)

Theorem: $\displaystyle\lim_{n\to\infty} \frac{\ell_n}{2\sqrt{n}} \overset{P}{=} 1$; $\displaystyle\lim_{n\to\infty} \frac{\ell_n - 2\sqrt{n}}{n^{1/6}} \overset{d}{=}$ Tracy-Widom distribution

Доклады Академии наук СССР
1977. Том 233, № 6

УДК 519.21 МАТЕМАТИКА

А. М. ВЕРШИК, С. В. КЕРОВ

АСИМПТОТИКА МЕРЫ ПЛАНШЕРЕЛЯ СИММЕТРИЧЕСКОЙ ГРУППЫ И ПРЕДЕЛЬНАЯ ФОРМА ТАБЛИЦ ЮНГА

(Представлено академиком А. Н. Колмогоровым 17 XII 1976)

Vershik, A. M.; Kerov, S. V.
Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. (English. Russian original)
Zbl 0406.05008
Sov. Math., Dokl. 18, 527-531 (1977); translation from Dokl. Akad. Nauk SSSR 233, 1024-1027 (1977).

On the distribution of the length of the longest increasing subsequence of random permutations

Authors: Jinho Baik, Percy Deift and Kurt Johansson
Journal: J. Amer. Math. Soc. **12** (1999), 1119-1178

Proof starts from $\ell_n \overset{d}{=} \lambda_1$ with

$$\mathrm{Prob}(\lambda) \sim \lim_{k\to\infty} \left[S_\lambda\left(\tfrac{1}{k}, \dots, \tfrac{1}{k}\right)\right]^2$$

Schur symmetric polynomial

partition $\lambda_1 \geq \lambda_2 \geq \dots \geq 0$ of n into integer summands

Motivation 5: spectral theory / integrable systems

Example: The Calogero-Sutherland Hamiltonian

$$H = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2} + \frac{\theta(1-\theta)}{2} \sum_{i<j} \frac{1}{\sin^2\left(\frac{x_i - x_j}{2}\right)}$$

Meaning: N quantum particles on a circle $0 \le x_i \le 2\pi$ with pairwise interactions.

Question: Eigenvalues / eigenfunctions of H?

They are: $\Psi_\lambda(x_1, \dots x_N) = J_\lambda(e^{2\pi i x_1}, \dots, e^{2\pi i x_N}) \cdot \prod_{i<j} \sin^\theta\left(\frac{x_i - x_j}{2}\right)$

For integers $\lambda_1 \ge \dots \ge \lambda_N$, $J_\lambda(z_1, \dots z_N)$ is Jack symmetric polynomial

$$H \Psi_\lambda = - \sum_{i=1}^{N} (\lambda_i + \theta(N-i))^2 \Psi_\lambda$$

Motivation 6: linear algebra

finite field with q elements

Example: we deal with matrices over $\mathbb{F}_q$

Take $m \times m$ strictly upper triangular matrix g

$$\begin{pmatrix} 0 & & & * \\ & 0 & & \\ & & 0 & \\ 0 & & & 0 \end{pmatrix}$$

(canonical)

its Jordan normal form is encoded by a partition λ of m

$\lambda_1 \geq \lambda_2 \geq \dots$ — sizes of blocks in JNF

Let $N_{\lambda,\mu}$ be # strictly upper triangular $(m+1)\times(m+1)$ matrices h, such that 1) $\mathrm{JNF}(h)=\mu$ 2) $m\times m$ corner of h $= g$

The law of large numbers and the central limit theorem for the jordan normal form of large triangular matrices over a finite field

A. M. Borodin

Journal of Mathematical Sciences **96**, 3455–3471(1999) | Cite this article

Exercise: compute $N_{\lambda,\mu}$.

Fact:

$$Q_\lambda(x_1,x_2,\dots;q^{-1})\cdot\sum_i x_i = \sum_\mu Q_\mu(x_1,x_2,\dots;q^{-1})\cdot N_{\lambda,\mu}\,\frac{q^{n(\lambda)-n(\mu)-m}}{1-q^{-1}}$$

version of Hall-Littlewood symmetric polinomials

simple

Motivations / applications of symmetric functions:

- combinatorics
- representation theory
- integral computations
- probability
- spectral theory
- linear algebra

. . .

- Knot theory
- enumerative geometry
- number theory
- stochastic PDEs
- theoretical physics

We need to learn how to work with them.

(we will!)