Lecture 3f - How are FRFs Measured?

Thursday, October 8, 2020 6:50 PM

$$f(t) = \mathrm{Re}(F e^{i\omega t}) \longrightarrow x(t) = \mathrm{Re}(\underline{X} e^{i\omega t})$$

$$\underline{X} = H F \longrightarrow H(\omega) = \frac{X(\omega)}{F(\omega)}$$

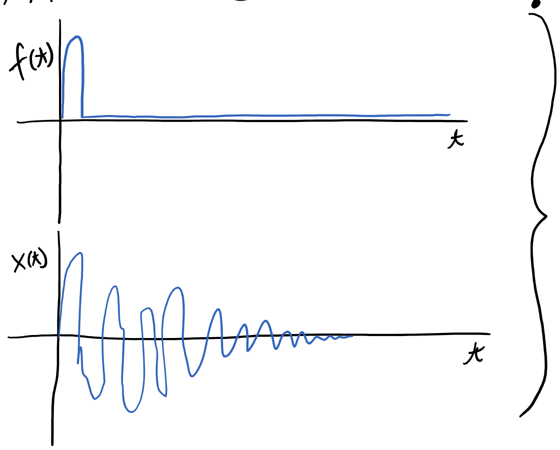
1.) apply $f(t)$ above
2.) Wait for steady-state
3.) Measure complex amplitudes $F, \underline{X}$
4.) $H = \underline{X}/F$

$\longrightarrow$ This method is great but can be very slow!

$$\tau = \frac{1}{\zeta \omega_n} \longrightarrow \tau = \frac{1}{0.01 \cdot (1\,\mathrm{Hz} \cdot 2\pi)} \hat{=} 16 \text{ sec}$$

$8\tau > 2$ min per frequency

Alternative: Use FFT!

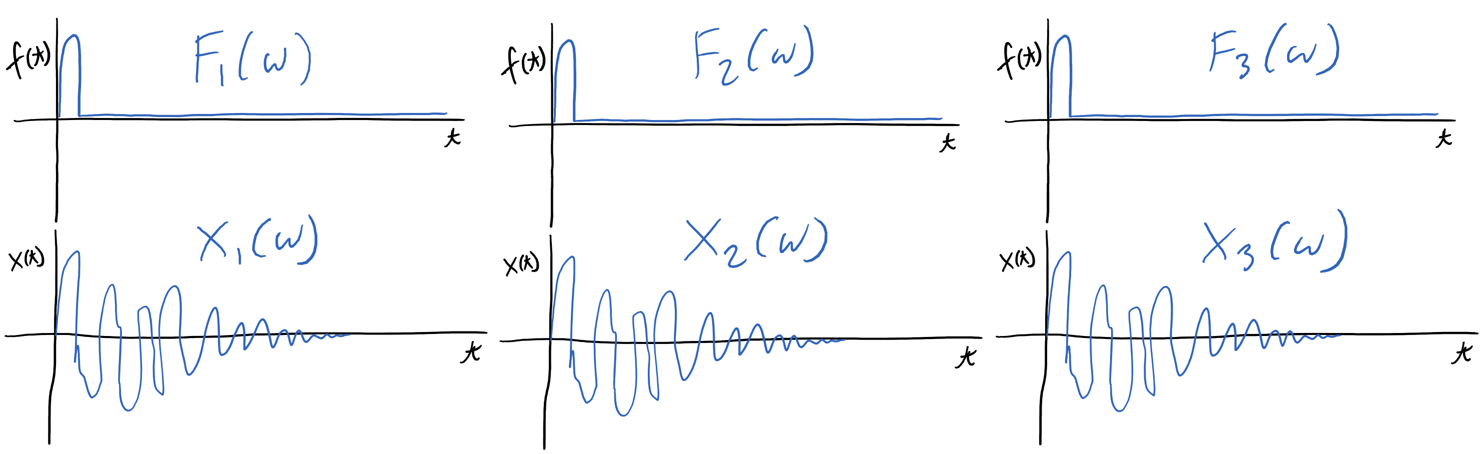


FFT gives $F(\omega)$, $X(\omega)$

for frequencies $[0, \omega_1, 2\omega_1, \cdots \omega_{max}]$

$$H(\omega) = \frac{X(\omega)}{F(\omega)}$$

Average over several measurements to minimize noise!



$$X_1(\omega) = H(\omega)\,F_1(\omega)$$
$$X_2(\omega) = H(\omega)\,F_2(\omega) \quad \Rightarrow \quad \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} H \qquad \text{at each freq. } (\omega)$$
$$X_3(\omega) = H(\omega)\,F_3(\omega)$$

"H_1" Solution

$$\begin{bmatrix} F_1^* & F_2^* & F_3^* \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} F_1^* & F_2^* & F_3^* \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} H$$

$$(F_1^* X_1 + F_2^* X_2 + F_3^* X_3) = \sum_{j=1}^{3} F_j^* X_j$$

$$S_{XF} = \frac{1}{N} \sum_{j=1}^{N} X_j F_j^*$$

$$S_{XF} = S_{FF} H$$

$$\boxed{H_1(\omega_K) = S_{FX}(\omega) S_{FF}^{-1}(\omega_K)}$$

"H_2" Solution

$$\begin{bmatrix} X_1^* & X_2^* & X_3^* \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} X_1^* & X_2^* & X_3^* \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} H$$

$$S_{XX} = S_{FX} H$$

$$\boxed{H_2(\omega) = S_{XX} S_{FX}^{-1}}$$

These formulas are also valid for a MIMO test

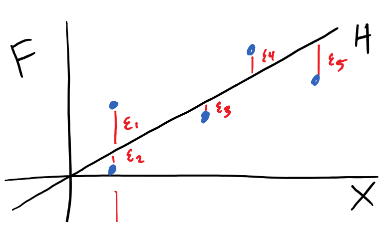
$$\Big[\underbrace{\{X(\omega)\}_1}_{N_o \times 1}\ \{X(\omega)\}_2 \cdots \{X(\omega)\}_{N_{avg}}\Big] = \underbrace{[H(\omega)]}_{N_o \times N_i} \Big[\underbrace{\{F(\omega)\}_1}_{N_i \times 1}\ \underbrace{\{F(\omega)\}_2 \cdots \{F(\omega)\}_{N_{avg}}}_{[F(\omega)]}\Big]$$

post-multiply by $[F(\omega)]^H$

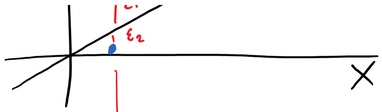
$$(\ )^H = \text{Hermetian} = ((\ )^T)^*$$

$$\underbrace{S_{XF}}_{N_o \times N_i} = \frac{1}{N} \sum_{j=1}^{N_{avg}} \{X(\omega)\}_j \{F(\omega)\}_j = \Big[\{X\}_1\ \{X\}_2 \cdots \{X\}_{N_{avg}}\Big] \begin{bmatrix} \{F\}_1^H \\ \{F\}_2^H \\ \vdots \\ \{F\}_{N_{avg}}^H \end{bmatrix} \frac{1}{N_{avg}}$$

Least Squares Solution

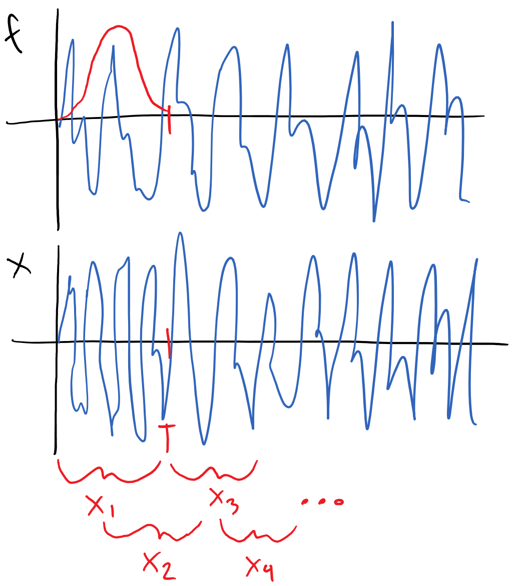


minimize $\left(\sum_j \varepsilon_j^2\right)$

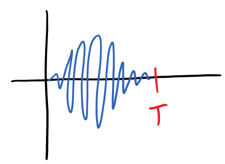


Simple Method: $H_j = \dfrac{X_j}{F_j} \rightarrow H = \dfrac{1}{N_{avg}} \sum_{j=1}^{N_{avg}} H_j$

But here $H_2 = \dfrac{X_2}{F_2} = \dfrac{X_2}{0} = \infty$ destroys average



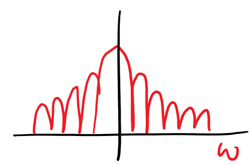
→ Apply window to minimize Leakage:



Hanning Window:

$$W(t) = \left(1 - \cos\left(\frac{\omega_1 t}{2}\right)\right)^2$$

$W(t) \rightarrow W(\omega)$



Distortion typically minimal if $T > 4\tau$