Copy 2: Lecture 3b - MDOF Forced Response

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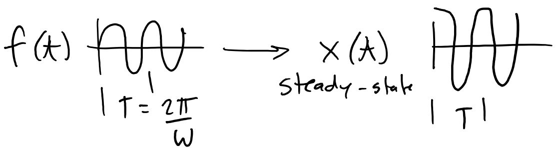
SDOF System

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

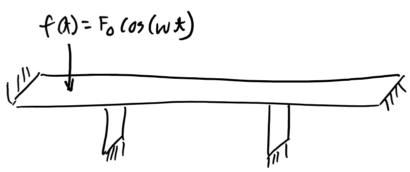


MDOF Systems:

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = Re\left(\{F\}e^{i\omega t}\right)$$

assume: $\{x\} = Re\left(\{X\}e^{i\omega t}\right)$

Direct solution: $\{X\} = \left(-\omega^2[M] + i\omega[C] + [K]\right)^{-1}\{F\}$



Modal Solution:

recall: $\left([K] - \omega^2[M]\right)\{\phi\} = 0 \longrightarrow \omega_1, \omega_2 \cdots \omega_N$, $[\Phi]$

$[\Phi]^T[K][\Phi] = \begin{bmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \end{bmatrix}$

$[\Phi]^T[M][\Phi] = [I]$

$\{x(t)\} = [\Phi]\{\eta(t)\} \longrightarrow$

$$\ddot{\eta}_j + 2\zeta_j\omega_j\dot{\eta}_j + \omega_j^2\eta_j = \{\Phi\}_j^T\{f(t)\}$$
$$= \{\Phi\}_j^T Re\left(\{F\}e^{i\omega t}\right)$$
$$= Re\left(F_{m,j}\, e^{i\omega t}\right)$$

as for SDOF System:

$$\eta_j(t) = Re\left(N_j e^{i\omega t}\right)$$

$$N_j = \frac{\{\Phi\}_j^T\{F\}}{\omega_j^2 - \omega^2 + i\omega 2\zeta_j\omega_j}$$

$\{x(t)\} = [\quad]\{Re(N_j e^{i\omega t})\}$

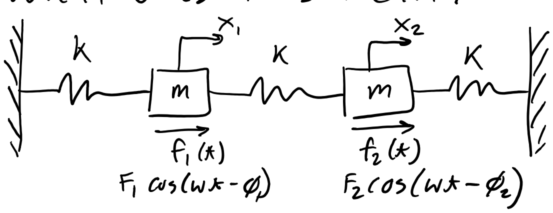
$$\omega_j^2 - \omega^2 + i\omega 2\zeta_j \omega_j$$

$$\begin{Bmatrix} X_1(t) \\ X_2(t) \\ \vdots \end{Bmatrix} = \begin{bmatrix} \{\Phi\}_1 & \{\Phi\}_2 & \cdots \end{bmatrix} \begin{bmatrix} Re(N_1 e^{i\omega t}) \\ Re(N_2 e^{i\omega t}) \\ \vdots \end{bmatrix}$$

$$\{x(t)\} = Re\left(\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \end{Bmatrix} e^{i\omega t}\right) \qquad \begin{Bmatrix} X_1 \\ X_2 \\ \vdots \end{Bmatrix} = \begin{bmatrix} \Phi \end{bmatrix} \begin{Bmatrix} N_1 \\ N_2 \\ \vdots \end{Bmatrix}$$

$$\{X\} = \sum_{j=1}^{N} \frac{\{\Phi\}_j \left(\{\Phi\}_j^T \{F\}\right)}{\omega_j^2 - \omega^2 + i\omega 2\zeta_j \omega_j} = [H(\omega)]\{F\}$$

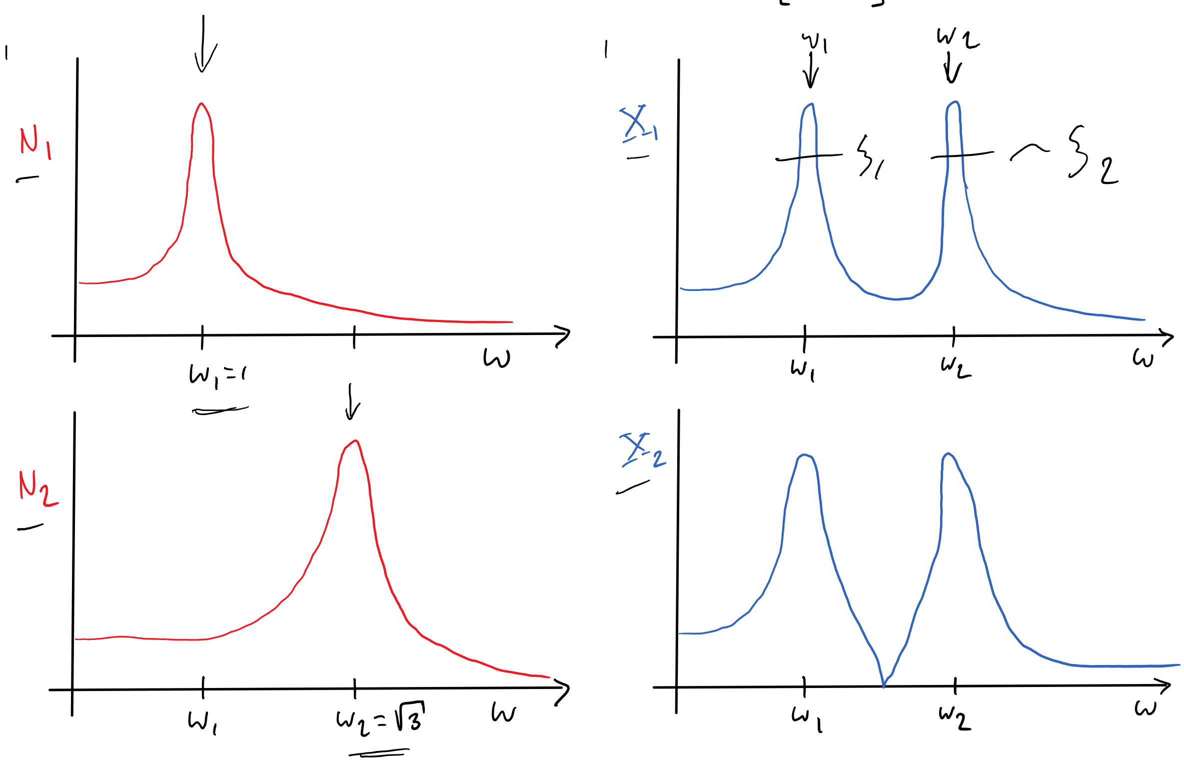
What does this mean?



$$\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

$(k = \frac{1}{2},\ m = \frac{1}{2})$

$$[\Phi] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \omega_1 = 1,\ \omega_2 = \sqrt{3}$$



Near $\omega = \omega_1$

$$\{X\} = \sum_{j=1}^{N} \frac{\{\Phi\}_j \left(\{\Phi\}_j^T \{F\}\right)}{\omega_j^2 - \omega^2 + i\omega 2\zeta_j \omega_j}$$

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = c\{\Phi\}_1$$

negligible

$$(\wedge) = \sum_{j=1} \frac{\quad}{\omega_j^2-\omega^2+i\omega 2\zeta_j\omega_j}$$

$$\text{small} \quad = \frac{\{\Phi\}_1\left(\{\Phi\}_1^T\{F\}\right)}{\cancel{\omega_1^2-\omega^2}+i\omega 2\zeta_1\omega_1} + \frac{\{\Phi\}_2\left(\{\Phi\}_2^T\{F\}\right)}{\omega_2^2-\omega^2+i\omega 2\zeta_2\omega_2} \text{ — large}$$

negligible

$\Rightarrow$ Experimental Modal Analysis EMA

BASIC — Advanced

1.) Measure FRFs
2.) Peaks $\longrightarrow$ nat. freqs ✓ — Least Squares
- width of peaks (half-power pts) $\longrightarrow$ damping ✓
- Peaks across many sensors : Mode shapes

$\omega_1 = \omega_2 \quad (\zeta_1 = \zeta_2)$

$$[H(\omega)] = \frac{\{\Phi\}_1\{\Phi\}_1^T}{\omega_1^2-\omega^2+i\omega 2\zeta_1\omega_1} + \frac{\{\Phi\}_2\{\Phi\}_2^T}{\omega_2^2-\omega^2+i\omega 2\zeta_2\omega_2}$$

$$[H(\omega)] = \frac{\{\phi\}_1\{\phi\}_1^T + \{\phi\}_2\{\phi\}_2^T}{\omega_1^2-\omega^2+i\omega 2\zeta_1\omega_1}$$ — Hard to see 2 modes

SIMO – Single-input multi-output

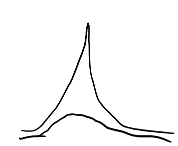
$$\{F\} = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \qquad \{X\} = \begin{Bmatrix} \bar{X}_1 \\ \bar{X}_2 \end{Bmatrix}$$

$$\{\phi\}_1^T\{F\} = [\Phi_{11} \quad \Phi_{21}]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \Phi_{11} = \Phi_{pr}$$

p = drive point $\quad r$ = mode #

$$\{H(\omega)\} = \frac{\{\Phi\}_1 \cancel{\{\Phi\}_1^T}\ \Phi_{p1}}{\omega_1^2-\omega^2+i\omega 2\zeta_1\omega_1} + \frac{\{\Phi\}_2 \cancel{\{\Phi\}_2^T}\ \Phi_{p2}}{\omega_2^2-\omega^2+i\omega 2\zeta_2\omega_2}$$

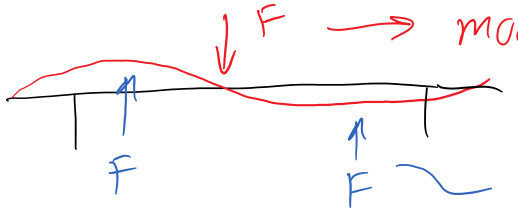
$$\{H(\omega)\}\Big|_{\omega \cong \omega_1} = c\{\Phi\}_1 \qquad c = \frac{\Phi_{p1}}{i\omega 2\zeta_1\omega_1}$$



$\Phi_{p1} \neq 0$ for mode to appear — light damping

$\perp F \longrightarrow$ mode missed

damping



mode missed

mode captured

MISO - Multi-input single output

meas at x_p $\quad \{F\} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ & & & \ddots \end{bmatrix}$

$$H_p\big|_{1st} = \frac{\phi_{pr}\{\phi\}_r^T}{den}\begin{bmatrix}1\\0\\0\end{bmatrix} = \phi_{pr}\,\phi_{1r}\,\frac{1}{den}$$

$$H_p\big|_{2nd} = \frac{\phi_{pr}\{\phi\}_r^T}{den}\begin{bmatrix}0\\1\\0\end{bmatrix} = \frac{\phi_{pr}\,\phi_{2r}}{den}$$

$$\left[H_p\big|_{1st} \quad H_p\big|_{2nd} \cdots\right] = \sum_{r=1}^{N} \frac{\phi_{pr}\left[\phi_{1r} \quad \phi_{2r} \cdots\right]}{\omega_r^2 - \omega^2 + j\omega 2\zeta_r\omega_r}$$

$= \text{Transpose SIMO}$

$$[H(\omega)] = \begin{bmatrix} \text{---} \; \text{MISO} \; \text{---} \\ \text{SIMO} \\ | \end{bmatrix} \qquad H = H^T$$

MIMO -

Meas all points / $\{F_1\} = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $\{F_2\} = \begin{Bmatrix}0\\1\\0\end{Bmatrix}$

$$[H(\omega)] = \sum_{r=1}^{N} \frac{[A_r]}{\omega_r^2 + \omega^2 + j\omega 2\zeta_r\omega_r}$$

$$[A_1] = \left[\{\phi\}_1\,\phi_{11} \quad \{\phi\}_1\,\phi_{21} \quad \cdots\right]$$

$\lfloor A_r \rfloor = \{\phi\}_r$ — F_1 on DOF 1 — F_2 on DOF 2 ..

└ Rank 1 matrix

Mode Indicator Function MIF

<u>CMIF</u> $\rightarrow [H(\omega)] = [U][\diagdown s \diagdown][V]^T$

Sing. Value Decomp SVD

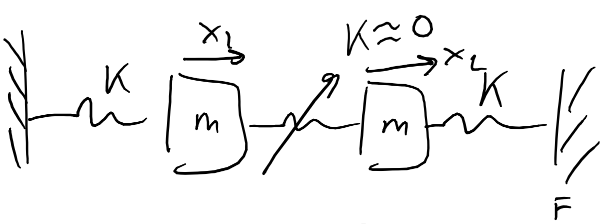
$[U]^T[U] = [I] \quad [V]^T[V] = [I]$

s_1, s_2, s_3

$s_1 > s_2,\ s_2 > s_3 ..$

$$\sum_j \{U\}_j \{V\}_j^T s_j$$

1 mode $s_1 \neq 0 \quad s_2 \cong s_3 \cong \dots \cong 0$



$\{\phi\}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \{\phi\}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

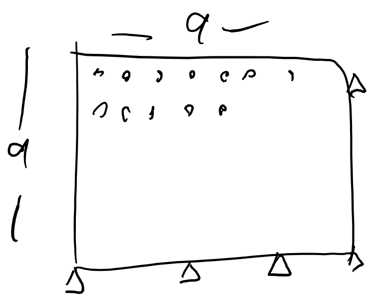
$\{H(\omega)\}$ SIMO $= \dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ \ 1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1 \ -1] \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{den} = \dfrac{\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}}{den}$

$= \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ mode shape

$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \ \ 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} [1 \ -1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

Mode shape seems to change w/ P.P.

MIMO $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = [A_r]$ rank = 2



SS plate

pairs of modes $\omega_j = \omega_n$ $j \neq n$