CHAPTER 2

TRANSIENT RESPONSE OF ONE-DEGREE-OF-FREEDOM SYSTEMS

Given the simplicity of a one-degree-of-freedom model, it might seem surprising that such a representation is widely used to explore real-world systems. In part, the significance of this model lies in the fact that it captures many of the fundamental physical phenomena manifested in vibrations of all systems. One-degree-of-freedom models also are important because modal analysis methods, which are essential to the study of complicated systems, convert multiple-degree-of-freedom systems to an equivalent set of one-degree-of-freedom systems.

A vibratory response might be a free vibration stemming from some set of initial conditions, or a forced response resulting from dynamic excitation. In this chapter we consider a variety of forces whose basic time signature changes with time, in some cases disappearing entirely. We refer to the response to such forces, as well as free vibration response, as *transient*, because the response we observe evolves as time elapses.

We begin our study by developing an extremely useful mathematical tool. Sinusoidal-like fluctuation is a common feature of many vibratory phenomena. The application of complex variable concepts substantially simplifies analytical and computational tasks involving sinusoidal functions.

2.1 HARMONIC FUNCTIONS

Harmonic time dependence is a synonym for sinusoidal variation. The term *harmonic* arises from music, where pure tones vary sinusoidally. In engineering applications, harmonic variation is a hallmark of alternating current and electromagnetic waves. Mechanical and structural systems are excited harmonically by rotating machinery, as we will see. Numerous other excitations may be represented by either a single harmonic term or a sum of such terms. Correspondingly, harmonic features of vibratory response arise in a variety of situations.

2.1.1 Basic Properties

When we say that a function is harmonic, or sinusoidal, it need not vary as a sine function. Figure 2.1 displays a typical harmonic function $u(t)$. Its mathematical form is

$$\boxed{u = A\sin(\omega t - \phi)} \tag{2.1.1}$$

The coefficient A is the *amplitude*. The *frequency* of u is ω; if t is measured in units of seconds, then ω has units of radians per second. The argument of a sinusoidal function, that is, $\omega t - \phi$, indicates the *phase*—for example, whether the sinusoidal

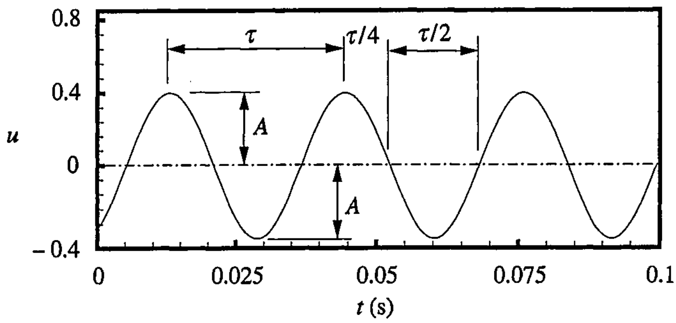


FIGURE 2.1 A typical harmonic function.

function is positive or negative, or whether it is close to a maximum or a zero. The *phase angle* is ϕ, which has units of radians.

The aforementioned parameters are manifested by the pattern in Figure 2.1. The amplitude is the peak excursion of u from the zero value, either positively or negatively, so that $A = \max|u|$. The frequency is directly related to the period τ, which is the time interval over which u repeats, such that $u(t + \tau) = u(t)$. The period appears in the figure as the time interval separating consecutive minima or maxima of u. If the pattern of u versus t is given, as it would be in an oscilloscope trace, then the most accurate value of τ would be obtained by measuring the time interval between adjacent zeroes, which would be $\tau/2$. This is so because it is difficult to identify precisely where a maximum occurs, but the zeroes are readily identified. To construct the relation between ω and τ, we observe that repetition of a sinusoidal function corresponds to an increase of the argument by 2π. The arguments at two instants separated by a period are $\omega t - \phi$ and $\omega(t + \tau) - \phi$. We take the difference and equate it to 2π, which leads to

$$\boxed{\omega = \frac{2\pi}{\tau}} \tag{2.1.2}$$

It is standard practice to describe the frequency in units of hertz (Hz), which represents the number of cycles (that is, periods) that occur in a one-second interval. If τ seconds are required for one period, then $1/\tau$ periods occur in one second. We shall use the symbol f to denote *cyclical frequency* measured in hertz, so we have

$$\boxed{f = \frac{1}{\tau} = \frac{\omega}{2\pi}} \tag{2.1.3}$$

Note that ω, rather than f, is the quantity to be used in any computations.

To understand the role of the phase angle consider the case where $\phi = 0$, so that the graph of u is a sine curve. The first zero would occur at $t = 0$, and the first maximum would occur at $t = \pi/2\omega$. When ϕ is nonzero, $u = 0$ when the phase $\omega t - \phi = 0$, which corresponds to $t = \phi/\omega$. Similarly, u has a maximum value when $\omega t - \phi = \pi/2$, which gives $t = \pi/2\omega + \phi/\omega$. Indeed, if $\phi > 0$, any feature of a sine function that occurs at instant t is displayed by u at a later time $t' = t + \phi/\omega$. The quantity ϕ/ω represents a time delay. We say that u *lags* relative to a sine by a time ϕ/ω, and ϕ is the *phase lag*. If ϕ were negative, we would say that u *leads* a sine, and $-\phi$ is the *phase lead*. It is common to describe a phase angle in degrees, but radians is the only acceptable unit for computations.

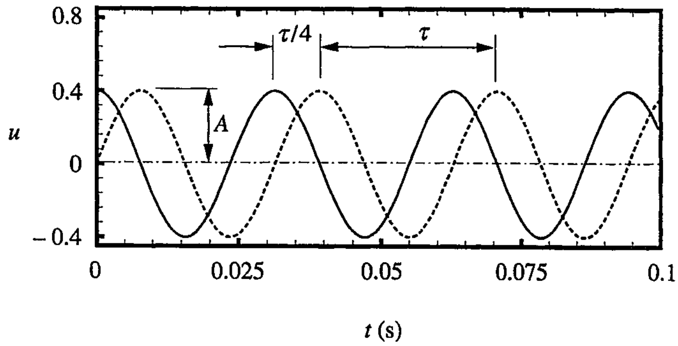


FIGURE 2.2 Phase delay of a sine relative to a cosine.

The phase angle is only meaningful if its reference function is specified. The reference is usually a sine or cosine function without a phase angle. We refer to such functions as a *pure sine* or *pure cosine*, respectively. For example, suppose that we wish to use a cosine function to describe $u(t)$ in Figure 2.1. The amplitude and period do not depend on whether we use a sine or cosine, so the plotted function would fit $u = A\cos(\omega t - \phi')$. The question is, What is ϕ' in terms of ϕ? A simple answer comes from matching the two forms at $t = 0$, which leads to $\sin(-\phi) = \cos(-\phi')$, that is, $-\sin(\phi) = \cos(\phi')$. This has multiple roots; we select $\phi' = \phi + \pi/2$. This choice is suggested by Figure 2.2, which shows that a sine function may be pictured as being delayed by $\tau/4$ relative to a cosine function. In other words, the phase lag of a sine relative to a cosine is 90°, or equivalently, the phase lead of a cosine relative to a sine is 90°. In a similar vein, we could say that the negative of a sine or cosine lags (or leads) the corresponding positive function by 180°.

EXAMPLE 2.1

Measurement of a harmonic function $F(t)$ leads to the observation that the maximum value of the function is 2 kN, that the elapsed time from a maximum to the first following zero value is 0.2 s, and that the earliest time $t > 0$ at which $F = 0$ and $\dot{F} > 0$ is 0.3 s. Determine the functional form of $F(t)$.

Solution This problem will enhance our familiarity with the fundamental properties of harmonic functions. We begin by noting that the elapsed time from maximum F to zero for a harmonic function is one quarter of the period, so $T = 4(0.2)$. Hence, the cyclical and circular frequencies are

$$f = \frac{1}{0.8} = 1.25 \text{ Hz} \quad \Rightarrow \quad \omega = 2.5\pi \text{ rad/s}$$

The standard form of a harmonic function is $F = A\sin(\omega t - \phi)$. The amplitude A is the maximum value, so $A = 2000$ N. To determine ϕ, we use the fact that the instant at which a harmonic function is zero and increasing corresponds to a zero value for the argument of the function. It is given that this instant is $t = 0.3$ s. Hence, it must be that

$$\omega(0.3) - \phi = 0 \quad \Rightarrow \quad \phi = 0.75\pi$$

The corresponding function is

$$F = 2000\sin(2.5\pi t - 0.75\pi) \text{ N}$$

2.1.2 Complex Variable Representation

The process of converting phase angles between sine and cosine functions is the first of many tasks that require the use of trigonometric identities. A more difficult one involves adding terms that have the same frequency, but different amplitudes and phase angles. We will use complex exponentials, rather than real functions, to describe harmonically varying quantities. Doing so will allow us to perform all operations with only a few identities. Furthermore, the use of complex exponentials will drastically simplify solving the differential equations of motion.

The foundation for the procedure is Euler's formula,

$$\exp(i\omega t) = \cos(\omega t) + i\sin(\omega t) \tag{2.1.4}$$

This follows from the definition of the cosine and sine functions in the complex plane,

$$\begin{aligned}\cos(\omega t) &= \frac{1}{2}[\exp(i\omega t) + \exp(-i\omega t)] = \frac{e^{i\omega t} + e^{-i\omega t}}{2}\\ \sin(\omega t) &= \frac{1}{2i}[\exp(i\omega t) - \exp(-i\omega t)] = \frac{e^{i\omega t} - e^{-i\omega t}}{2i}\end{aligned} \tag{2.1.5}$$

These definitions may always be used to replace the trigonometric functions. However, it is awkward and repetitive to carry around the second part of each definition, because it is merely the complex conjugate of the first part. For most operations, it is simpler to extract the desired function from eq. (2.1.4). When we wish to extract the cosine function from the complex exponential in eq. (2.1.4), we take the real part. To extract the sine function, we could extract the imaginary part, but doing so might lead to difficulties if we must combine terms, some of which are real parts and others are imaginary parts. We therefore will *make it standard practice to always use real parts.* Thus, to extract the sine function, we divide the complex exponential by i, and then take the real part. In other words

$$\boxed{\begin{aligned}\cos(\omega t) &= \mathrm{Re}[\exp(i\omega t)] = \mathrm{Re}\left(e^{i\omega t}\right)\\ \sin(\omega t) &= \mathrm{Re}\left[\frac{1}{i}\exp(i\omega t)\right] \equiv \mathrm{Re}[-i\exp(i\omega t)] = \mathrm{Re}\left(-i e^{i\omega t}\right)\end{aligned}} \tag{2.1.6}$$

Equivalent forms based on using $\exp(-i\omega t)$ are used by some practitioners. For this reason it is important to examine any treatment using complex functions to ascertain which convention (plus or minus sign) has been adopted.

Now consider the function u described by eq. (2.1.1). The argument of the sine must be the argument of the complex exponential, and the amplitude A is real, so it may be brought inside the bracket. We therefore have

$$u = A\sin(\omega t - \phi) = \mathrm{Re}\left[\frac{A}{i}\exp(i\omega t - i\phi)\right] \tag{2.1.7}$$

We now use the property that the exponential of a sum is the product of the individual exponentials to rewrite the foregoing as

$$u = \mathrm{Re}\left\{\frac{A}{i}\exp(-i\phi)\exp(i\omega t)\right\} \tag{2.1.8}$$

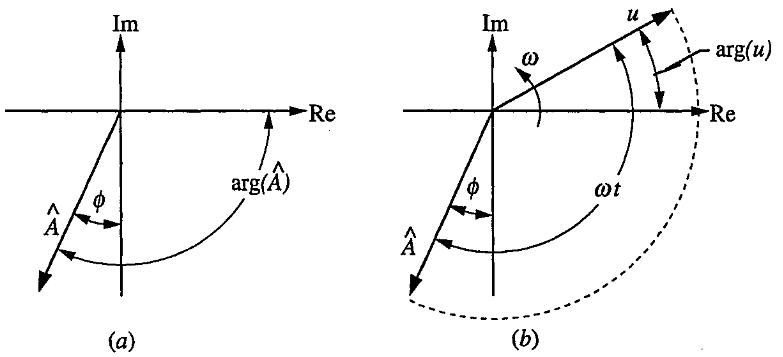


FIGURE 2.3 (*a*) Complex amplitude of a sine function with phase lag ϕ. (*b*) Complex plane representation of a sine function with phase lag ϕ.

The factor multiplying $\exp(i\omega t)$ is a complex constant, which we call the *complex amplitude,* $\hat{A}$. To plot this quantity in the complex plane, we use the polar representation. Because the polar form of i is $\exp(i\pi/2)$, we have

$$u = \mathrm{Re}[\hat{A}\exp(i\omega t)], \quad \hat{A} = \frac{A}{i}\exp(-i\phi) = A\exp\left[-i\left(\phi + \frac{\pi}{2}\right)\right] \tag{2.1.9}$$

Thus, the magnitude of $\hat{A}$ is A, and the polar angle of $\hat{A}$, which is called the *argument,* is $\phi + \pi/2$ below the positive real axis, as shown in Figure 2.3(*a*),

$$|\hat{A}| = A, \qquad \arg(\hat{A}) = -\phi - \frac{\pi}{2} \tag{2.1.10}$$

To construct the representation of u in the complex plane, we note that $|\exp(i\omega t)| = 1$, which means that we obtain u by merely adding the argument ωt to the argument of $\hat{A}$, as shown in Figure 2.3(*b*). This construction shows that using complex variables to represent a harmonic function at any instant is equivalent to depicting it as a rotating vector in the complex plane. Because the value of ωt increases linearly with elapsed time, the vector representing a complex function rotates counterclockwise at angular speed ω as t increases.

EXAMPLE 2.2

Write the following functions as complex exponentials. Express the corresponding complex amplitude in polar and rectangular form.

$$F = 5\cos(50t + 0.4), \quad G = 20\sin(10t - 0.5), \quad H = 30\sin\left(400t - \frac{2\pi}{3}\right)$$

Solution This exercise highlights the steps by which we convert harmonic functions to complex form. The basic idea is to apply the representations of sine and cosine functions in terms of complex functions, as given by eqs. (2.1.6), and then to match the converted form to the standard complex representation of a harmonic function,

$$F = \mathrm{Re}[A\exp(i\omega t)]$$

For the first function, we use the fact that a cosine is the real part of a complex exponential function, so that

$$F = 5\cos(50t + 0.4) = \mathrm{Re}\{5\exp[i(50t + 0.4)]\}$$
$$= \mathrm{Re}[5\exp(0.4i)\exp(i50t)]$$

We match this to the standard form, from which we conclude that the complex amplitude, A, is

+1.947i

$$A_F = 5\exp(0.4i) = 4.605 + 1.9\cancel{5}47i$$

A sine function is the imaginary part of a complex exponential, so the second function is

$$G\ \cancel{X} = 20\sin(10t - 0.5) = \mathrm{Re}\left\{\frac{20}{i}\exp[i(10t - 0.5)]\right\}$$
$$G = \mathrm{Re}\left[\frac{20}{i}\exp(-0.5i)\exp(i10t)\right]$$

This matches the standard representation of a harmonic function if the complex amplitude is

$$A_G = \frac{20}{i}\exp(-0.5i) = 20\exp(-0.5\pi i - 0.5i) = -9.589 - 17.552i$$

We follow the same procedure for the third of the given functions,

$$H\ \cancel{X} = 30\sin\left(400t - \frac{2\pi}{3}\right) = \mathrm{Re}\left\{\frac{30}{i}\exp\left[i\left(400t - \frac{2\pi}{3}\right)\right]\right\}$$
$$H = \mathrm{Re}\left[\frac{30}{i}\exp\left(-\frac{2\pi}{3}i\right)\exp(i400t)\right]$$

Matching this to the standard form gives

$$A_H = \frac{30}{i}\exp\left(-\frac{2\pi}{3}i\right) = 30\exp\left(-0.5\pi i - \frac{2\pi}{3}i\right) = -25.98 + 15i$$

As a sidebar to assist readers who are not comfortable with performing computations with complex numbers, let us review a few fundamental techniques. Many of the operations can be implemented directly on a calculator that recognizes complex numbers, but software like MATLAB and Mathcad offers an advantage, in that it retains a record of what one has done. In MATLAB, complex constants may be entered by writing them in the conventional manner, for example, $3 + 4i$ or $3 + 4/i$. If we wish to use variables in a similar manner, multiplication by i requires a multiplication sign, for example, $x + y*i$. Similarly, complex exponentials are obtained by following the written form, using the "exp" function. Complex conjugates come into play in some circumstances. The best strategy here is to use the "conj" function in MATLAB, rather than the prime operator ($'$), which also performs a transpose when applied to matrices. An important operation is conversion of complex numbers between polar and rectangular form. The latter is the internal format of a complex number. To find the magnitude of a complex number z, we write abs ~~*mag*~~ (z), while *angle* (z) gives the polar angle in radians relative to the positive real axis. Note that radians is also the angle measure that must be used for the exponential function.

Most of the preceding considerations also apply to Mathcad. One difference is that i must be accompanied by a numerical factor in all contexts if it is to be interpreted as $i = \sqrt{-1}$. Thus, we would write $x + y*1i$. Another difference is that the operation of finding the polar form of a complex number is achieved by writing $|z|$ to determine the magnitude, and *arg*(z) to obtain the polar angle.

2.1.3 Algebraic Operations

The complex exponential form simplifies many operations involving harmonic functions. For example, consider the earlier situation where we were given $u = A\sin(\omega t - \phi)$, and we wished to convert the expression to $u = A\cos(\omega t - \phi')$. Using a complex exponential to represent each form leads to

$$\frac{1}{i}\exp(i\omega t - i\phi) = \exp(i\omega t - i\phi') \tag{2.1.11}$$

Recall that $1/i = \exp(-i\pi/2)$. The equality must apply at all instants t, so we may cancel the common factor $\exp(i\omega t)$ on both sides. This yields

$$\exp(-i\pi/2)\exp(-i\phi) = \exp(-i\phi') \quad \Rightarrow \quad \phi' = \phi + \frac{\pi}{2} \tag{2.1.12}$$

which matches what we had deduced using real functions.

The notion that a complex function may be represented as a vector suggests that different functions at the same frequency may be added as vectors, based on a pictorial representation of the parallelogram law. Rather than doing so, we shall follow an algebraic procedure, which factors out the shared $\exp(i\omega t)$ dependence. For example, consider representing a harmonic function u that is known to be the sum of two other harmonics at the same frequency,

$$\begin{aligned} u &= \mathrm{Re}[\hat{A}\exp(i\omega t)] \\ &= A_1\sin(\omega t - \phi_1) + A_2\cos(\omega t + \phi_2) \end{aligned} \tag{2.1.13}$$

To determine the value of the complex amplitude $\hat{A}$ given A_1, A_2, ϕ_1, and ϕ_2, we use eqs. (2.1.6) to represent the sine and cosine as complex exponentials,

$$\begin{aligned} \mathrm{Re}[\hat{A}\exp(i\omega t)] &= \mathrm{Re}\left[\frac{A_1}{i}\exp(-i\phi_1)\exp(i\omega t)\right] + \mathrm{Re}[A_2\exp(i\phi_2)\exp(i\omega t)] \\ &= \mathrm{Re}\left\{\left[\frac{A_1}{i}\exp(-i\phi_1) + A_2\exp(i\phi_2)\right]\exp(i\omega t)\right\} = \mathrm{Re}\left\{\left[-iA_1 e^{-i\phi_1} + A_2 e^{i\phi_2}\right]e^{i\omega t}\right\} \end{aligned} \tag{2.1.14}$$

In order for the real parts to match at all t, the complex coefficients of $\exp(i\omega t)$ must match, which leads to

$$\hat{A} = \frac{A_1}{i}\exp(-i\phi_1) + A_2\exp(i\phi_2) \tag{2.1.15}$$

If we have values for each A_j and ϕ_j, we may evaluate $\hat{A}$ numerically using a calculator or computer software. If the quantities are algebraic, we proceed by converting the terms in the preceding from polar to rectangular form,

$$\begin{aligned} \hat{A} &= \frac{A_1}{i}[\cos(\phi_1) - i\sin(\phi_1)] + A_2[\cos(\phi_2) + i\sin(\phi_2)] \\ &= [-A_1\sin(\phi_1) + A_2\cos(\phi_2)] + i[-A_1\cos(\phi_1) + A_2\sin(\phi_2)] \end{aligned} \tag{2.1.16}$$

If we wish, we may convert this rectangular representation of $\hat{A}$ to polar form using $\hat{A} = A\exp(-i\phi) = A\cos(\phi) - iA\sin(\phi)$. Matching real and imaginary parts of the two forms for $\hat{A}$ leads to

$$\begin{aligned} A\cos(\phi) &= -A_1\sin(\phi_1) + A_2\cos(\phi_2) \\ -A\sin(\phi) &= -A_1\cos(\phi_1) + A_2\sin(\phi_2) \end{aligned} \tag{2.1.17}$$

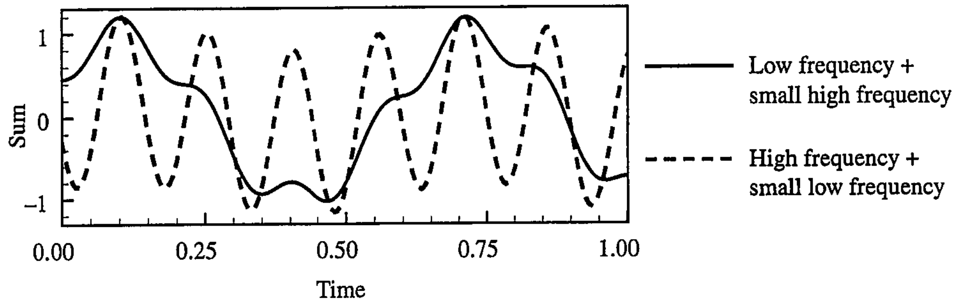


FIGURE 2.4 Summation of two harmonic functions at different frequencies.

We may determine the magnitude A of the complex amplitude by summing the squares of each of the above, while their ratio gives tan (ϕ). Note that in applying the arctangent function to the latter, we must be careful to place ϕ in the quadrant that is consistent with each of eqs. (2.1.17). Also note that the form of u corresponding to the polar representation of $\hat{A}$ is $u = \mathrm{Re}[A\exp(i\omega t - i\phi)]$, which means that A is the (real) amplitude and ϕ is the phase lag of u relative to a cosine function.

Many situations lead to summations of harmonic terms having different frequencies. Let us consider the case where two harmonics at frequencies ω_1 and ω_2 are combined. Two typical situations are shown in Figure 2.4. If $\omega_2 \gg \omega_1$ and the amplitude of the higher frequency component is much smaller than that of the lower frequency component, the sum appears to be the lower frequency term with a superimposed high-frequency fluctuation. On the other hand if the low-frequency component has the smaller amplitude, then the alteration of the high-frequency signal appears to result in slow fluctuation of the amplitude of the high-frequency signal.

In general, not much can be done to simplify the mathematical representation of a combination of two harmonics. An exception, which we will encounter later in this chapter, arises when the amplitudes of the two terms are equal. Let us consider the sum of two harmonic functions at frequencies ω_1 and ω_2, whose phase lags relative to cosine functions are ϕ_1 and ϕ_2, respectively. We replace each harmonic function with its definition in terms of complex exponentials. Note that in this step, we do not use the real part notation because we wish to combine terms at different frequencies. Thus, we write

$$\begin{aligned} u &= A\cos(\omega_1 t - \phi_1) + A\cos(\omega_2 t - \phi_2) \\ &= \tfrac{1}{2}A\{\exp[i(\omega_1 t - \phi_1)] + \exp[-i(\omega_1 t - \phi_1)]\} \\ &\quad + \tfrac{1}{2}A\{\exp[i(\omega_2 t - \phi_2)] + \exp[-i(\omega_2 t - \phi_2)]\} \end{aligned} \tag{2.1.18}$$

To combine these terms we define ω_{av} and ϕ_{av} to be the average frequency and phase lag, while Δ_ω and Δ_ϕ are the deviations from the average values,

$$\begin{aligned} \omega_{\mathrm{av}} &= \tfrac{1}{2}(\omega_1 + \omega_2), \qquad & \Delta_\omega &= \tfrac{1}{2}(\omega_2 - \omega_1) \\ \phi_{\mathrm{av}} &= \tfrac{1}{2}(\phi_1 + \phi_2), \qquad & \Delta_\phi &= \tfrac{1}{2}(\phi_2 - \phi_1) \end{aligned} \tag{2.1.19}$$

We use these definitions to replace the absolute frequencies and phase angles in eq. (2.1.18). For instance, we have $\omega_1 = \omega_{\mathrm{av}} - \Delta\omega$ and $\omega_2 = \omega_{\mathrm{av}} + \Delta_\omega$. After some manipulation, we find that

$$u = \tfrac{1}{2}A\exp[i(\omega_{av}t - \phi_{av})]\{\exp[-i(\Delta_\omega t - \Delta_\phi)] + \exp[+i(\Delta_\omega t - \Delta_\phi)]\}$$
$$+ \tfrac{1}{2}A\exp[-i(\omega_{av}t + \phi_{av})]\{\exp[+i(\Delta_\omega t - \Delta_\phi)] + \exp[-i(\Delta_\omega t - \Delta_\phi)]\} \tag{2.1.20}$$

The terms within each pair of braces are the same complex representation of a cosine function at frequency Δ_ω. Factoring them out leaves the complex representation of a harmonic function at frequency ω_{av}. Hence, we have

$$u = 2A\cos(\Delta_\omega t - \Delta_\phi)\cos(\omega_{av}t - \phi_{av}) \tag{2.1.21}$$

Because $\omega_{av} > \Delta_\omega$, we interpret the above as varying harmonically at frequency ω_{av} with an amplitude $2A\cos(\Delta_\omega t - \Delta_\phi)$ that varies more slowly at frequency Δ_ω. We say that $u = \pm 2A\cos(\Delta_\omega t - \Delta_\phi)$ is the *envelope* function. When ω_1 and ω_2 are quite close, this combination is called a *beating signal.* It is readily produced musically by slightly mistuning one instrument relative to another, and then playing them with nearly equal intensities. A typical beating signal is displayed in Figure 2.5.

The interval π/Δ_ω over which the signal gets larger and then dies out is the *beat period.* Within each beat, the signal fluctuates at a frequency of ω_{av}, so the interval between zeroes is π/ω_{av}. The interval between successive minima or maxima of the signal is approximately $2\pi/\omega_{av}$.

In general, a beating signal is not periodic. The condition of periodicity requires that within the period T of the signal, each term contributing to the signal repeat an integer number of times. If any signal contains two harmonics, this requirement is $T = m(2\pi/\omega_1) = n(2\pi/\omega_2)$, where m and n are integers. This leads to $m/n = \omega_1/\omega_2$, which is possible only if ω_1/ω_2 is a rational fraction.

In the foregoing we found that a sum of two harmonic functions at different frequencies may be represented alternatively as a product. Occasionally, we need to go in the opposite direction by decomposing a product of harmonic functions into its individual components. It is imperative in such an operation to avoid a common mistake. The real part of a product of complex variables is not the product of the individual real parts, that is, if $u_1 = \text{Re}(z_1)$ and $u_2 = \text{Re}(z_2)$, then $u_1u_2 \neq \text{Re}(z_1z_2)$. Some treatments switch to real variables to handle the product, but that would require application of trigonometric identities. The complex definitions of the sine and cosine in eqs. (2.1.5) lead to the result directly. Equations

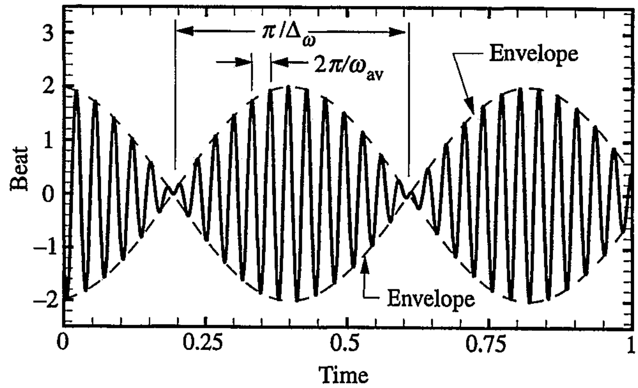


FIGURE 2.5 A typical beating signal.

(2.1.5) describe each real function as the sum of a complex exponential z and its complex conjugate z^*, $u_j = \frac{1}{2}(z_j + z_j^*)$. We then have

$$\begin{aligned} u_1 u_2 &= \tfrac{1}{4}(z_1 + z_1^*)(z_2 + z_2^*) \equiv \tfrac{1}{4}(z_1 + z_1^*)z_2 + \tfrac{1}{4}[(z_1 + z_1^*)z_2]^* \\ &\equiv \tfrac{1}{2}\mathrm{Re}[(z_1 + z_1^*)z_2] \equiv \tfrac{1}{2}\mathrm{Re}[z_1(z_2 + z_2^*)] \end{aligned} \tag{2.1.22}$$

As a verification for this procedure let us consider $\sin(\omega_1 t)\cos(\omega_2 t)$. With the aid of eqs. (2.1.5), we have

$$\begin{aligned} \sin(\omega_1 t)\cos(\omega_2 t) &= \frac{1}{2i}[\exp(i\omega_1 t) - \exp(-i\omega_1 t)]\frac{1}{2}[\exp(i\omega_2 t) + \exp(-i\omega_2 t)] \\ &= \frac{1}{2}\mathrm{Re}\left\{\frac{1}{i}[\exp(i\omega_1 t) - \exp(-i\omega_1 t)]\exp(i\omega_2 t)\right\} \\ &= \frac{1}{2}\mathrm{Re}\left\{\frac{1}{i}\exp[i(\omega_1 + \omega_2)t] - \frac{1}{i}\exp[i(-\omega_1 + \omega_2)t]\right\} \\ &= \frac{1}{2}\{\sin[(\omega_1 + \omega_2)t] - \sin[(-\omega_1 + \omega_2)t]\} \end{aligned} \tag{2.1.23}$$

It is not difficult to verify that the last form is equivalent to an identity associated with the sine of the sum of two angles.

An overview of the development thus far shows that Euler's formula and the fundamental rules of algebra are sufficient to manipulate the complex exponential representation of harmonic functions. Computations are readily implemented with scientific-type calculators and mathematical software. Most operations are done more directly than they would be if real functions were used.

EXAMPLE 2.3

A signal is measured to be $v = 12\sin(25t - 4.5)$. Decompose this signal into parts that are purely cosine and sine functions.

Solution This exercise will enhance our proficiency in using complex representations of harmonic functions. We recall that a sine is the imaginary part of a complex exponential to write

$$\begin{aligned} v = 12\sin(25t - 4.5) &= \mathrm{Re}\left\{\frac{12}{i}\exp[i(25t - 4.5)]\right\} \\ &= \mathrm{Re}\left[\frac{12}{i}\exp(-4.5i)\exp(i25t)\right] \end{aligned}$$

We now replace all polar forms of a complex quantity with their equivalent rectangular forms and combine real and imaginary parts, such that

$$\begin{aligned} v &= \mathrm{Re}\left[\frac{12}{i}(-0.2108 + 0.97753i)\exp(i25t)\right] \\ &= \mathrm{Re}\{(11.731 + 2.530i)[\cos(25t) + i\sin(25t)]\} \\ &= 11.731\cos(25t) - 2.530\sin(25t) \end{aligned}$$

EXAMPLE 2.4

Two harmonic functions are known to be $u_1 = 3\sin(40t)$ and $u_2 = 4\cos(40t + \pi/4)$. Express $u = u_1 - u_2$ as (a) a cosine function with a phase angle and (b) a sine function with a phase angle.

Solution In addition to illustrating the basic operations, this exercise will improve our ability to perform computations with complex numbers. Regardless of the form we wish for the final result, we begin by converting the given real functions into complex form. Thus, we write

$$u_1 = \mathrm{Re}\left[\frac{3}{i}\exp(i40t)\right]$$

$$u_2 = \mathrm{Re}\left\{4\exp\left[i\left(40t + \frac{\pi}{4}\right)\right]\right\} = \mathrm{Re}\left[4\exp\left(i\frac{\pi}{4}\right)\exp(i40t)\right]$$

We take the difference of the terms and collect the coefficients of exp $(i40t)$,

$$u = u_1 - u_2 = \mathrm{Re}\left\{\left[\frac{3}{i} - 4\exp\left(i\frac{\pi}{4}\right)\right]\exp(i40t)\right\}$$

For the sake of completeness, we shall explicitly display the arithmetic operations to simplify the coefficient. The reader is invited to perform the same operations solely with a calculator and with mathematical software. We convert each complex number from polar to rectangular form and combine like parts according to

$$u = \mathrm{Re}\left\{\left[-3i - 4\cos\left(\frac{\pi}{4}\right) - 4\sin\left(\frac{\pi}{4}\right)i\right]\exp(i40t)\right\}$$

$$= \mathrm{Re}[(-2.828 - 5.828i)\exp(i40t)]$$

How we proceed now depends on how u is to be represented. For a cosine function, the complex amplitude is the coefficient of exp $(i40t)$. We convert this coefficient to polar form, which leads to

$$u = \mathrm{Re}\{[-2.828 - 5.828i]\exp(i40t)\} = \mathrm{Re}\{6.478\exp(-2.023i)\exp(i40t)\}$$
$$= 6.478\cos(40t - 2.023)$$

Note that the phase angle is in the third quadrant because both the real and imaginary parts of the complex amplitude are negative.

When we wish to represent u as a sine function, we use the identity $\sin(z) = \mathrm{Re}\{(1/i)\exp(iz)\}$. To place the complex representation of u into real form, we place a factor i in the denominator, and introduce a compensating factor i into the complex amplitude. In addition, we use the fact that $i = \exp(i\pi/2)$. Hence,

$$u = \mathrm{Re}\left[(-2.828 - 5.828i)i\frac{\exp(i40t)}{i}\right] = \mathrm{Re}\left[(5.828 - 2.828i)\frac{\exp(i40t)}{i}\right]$$

$$= \mathrm{Re}\left[6.478\exp(-0.4518i)\frac{\exp(i40t)}{i}\right] = 6.478\sin(40t - 0.4518)$$

2.2 FREE VIBRATION

A one-degree-of-freedom system may be represented by a standard mass-spring-dashpot system. The mechanical properties of such a system are the inertia, stiffness, and damping coefficients corresponding to the generalized coordinate we

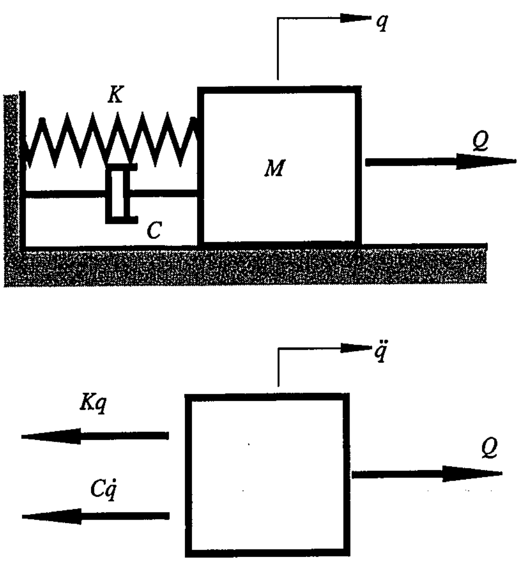


FIGURE 2.6 Standard one-degree-of-freedom system.

have selected. Because there is only one of each coefficient, we may dispense with subscripts, so the displacement q is affected by the mass M, spring K, and dashpot C. This leads to the generic one-degree-of-freedom system in Figure 2.6. Note that it is conventional to depict q as translational displacement of a small block that slides over a smooth horizontal surface, so the spring and dashpot are extensional types. In the event that q is an angular position quantity, M would be a moment of inertia, and the spring and dashpot would be rotational types. Depending on the type of quantity q represents, the generalized force Q exciting the system is either a force or a moment.

The equation of motion is the single differential equation,

$$M\ddot{q} + C\dot{q} + Kq = Q(t) \tag{2.2.1}$$

In general, the equation of motion governs the acceleration subsequent to the time when the motion is initiated. We usually set $t = 0$ as the initial time. Correspondingly, we must specify the initial displacement and velocity at $t = 0$, which we denote as q_0 and $\dot{q}_0$, respectively. Once the equation of motion and initial conditions are stated, the problem is well posed and ready to be solved.

The first type of response we consider is *free vibration*, which means that there are no external excitations, $Q \equiv 0$. The motion in this case results from the initial conditions, which state that the system was not initially at rest in its static equilibrium position. Because the right side is zero the standard equation of motion is a homogeneous linear differential equation. All solutions of this equation depend exponentially on time, so we seek a homogeneous solution in the form

$$q = B\exp(\lambda t) \tag{2.2.2}$$

where the constants B and λ must be determined. We find an equation for λ by substituting the trial solution into the homogeneous equation of motion. Because the equation must be satisfied at all t, we may factor out $B\exp(\lambda t)$ from the substituted form, which leads to a *characteristic equation* for λ, specifically

$$\boxed{M\lambda^2 + C\lambda + K = 0} \tag{2.2.3}$$

This is a quadratic equation, so there are two values of λ. Correspondingly, there are two homogeneous solutions. The nature of these solutions depends on the relative values of K, M, and C.

2.2.1 Undamped Systems

Dissipation is absent in an ideal system, which we model by setting $C = 0$. The roots of the characteristic equation in this case are purely imaginary, being given by

$$\boxed{\lambda = \pm i\omega_{\text{nat}}, \qquad \omega_{\text{nat}} = \sqrt{\frac{K}{M}}} \tag{2.2.4}$$

The parameter ω_{nat} is the *natural frequency*. (The reason for this term will soon become apparent.) It is a fundamental property of the system that will appear in most aspects of the response.

Because the characteristic roots are imaginary, there are two corresponding complex exponential solutions. The coefficient B associated with each may be complex, and they need not be the same. Hence, the general solution is

$$q = B_1 \exp(i\omega_{\text{nat}} t) + B_2 \exp(-i\omega_{\text{nat}} t) \tag{2.2.5}$$

This solution has a complex form, but q is a real quantity. To resolve this dilemma we observe that the two exponential functions are complex conjugates. If the coefficients are also complex conjugates, then the imaginary parts will cancel. We therefore set $B_1 = \frac{1}{2}\hat{A}$ and $B_2 = \frac{1}{2}\hat{A}^*$, where we introduce the half factor to simplify the final form of q. The corresponding solution is

$$\boxed{q = \tfrac{1}{2}\hat{A}\exp(i\omega_{\text{nat}} t) + \tfrac{1}{2}\hat{A}^* \exp(-i\omega_{\text{nat}} t) = \operatorname{Re}[\hat{A}\exp(i\omega_{\text{nat}} t)]} \tag{2.2.6}$$

In other words the free response of an undamped one-degree-of-freedom system is a harmonic motion that occurs at the natural frequency of the system.

The complex amplitude is dictated by the initial conditions. To determine this quantity we set $\hat{A} = c_1 - ic_2$, where c_1 and c_2 are real and the minus sign is a matter of convenience. This leads to $q = \operatorname{Re}[(c_1 - ic_2)\exp(i\omega_{\text{nat}} t)]$, which reduces to

$$\boxed{q = c_1 \cos(\omega_{\text{nat}} t) + c_2 \sin(\omega_{\text{nat}} t)} \tag{2.2.7}$$

In order to satisfy the initial conditions, we form $\dot{q}$ by differentiating the preceding with respect to t, and then evaluate q and $\dot{q}$ at $t = 0$, which leads to

$$\boxed{c_1 = q_0, \quad c_2 = \frac{\dot{q}_0}{\omega_{\text{nat}}} \quad \Rightarrow \quad q = q_0 \cos(\omega_{\text{nat}} t) + \frac{\dot{q}_0}{\omega_{\text{nat}}} \sin(\omega_{\text{nat}} t)} \tag{2.2.8}$$

In a graph, the initial displacement q_0 is the intercept with the axis $t = 0$ of the curve showing q as a function of t. The slope of this curve is $\dot{q}$, so the initial velocity $\dot{q}_0$ is the slope at that intercept.

The preceding is the real form of the free vibration solution. The results may be converted to complex form. Using the polar form to represent $\hat{A} = c_1 - ic_2 = A\exp(-i\phi)$ leads to

$$A\cos(\phi) = q_0, \quad A\sin(\phi) = \frac{\dot{q}_0}{\omega_{\text{nat}}} \tag{2.2.9}$$

from which we find that the amplitude and phase angle are

$$\boxed{\begin{gathered} q = A\cos(\omega_{\text{nat}} t - \phi) \\ A = \left[q_0^2 + \left(\frac{\dot{q}_0}{\omega_{\text{nat}}} \right)^2 \right]^{1/2}, \qquad \phi = \tan^{-1}\left(\frac{\dot{q}_0}{\omega_{\text{nat}} q_0} \right) \end{gathered}} \tag{2.2.10}$$

Note that evaluation of the arctangent requires that the quadrant be consistent with the values of $A\cos(\phi)$ and $A\sin(\phi)$.

The oscillatory nature of an undamped free vibration could have been predicted by using physical arguments. In the absence of dissipation, the system is conservative. The mechanical energy $T + V$ is therefore constant, equaling the value set by the initial conditions. When the system passes the static equilibrium position, $q = 0$, the potential energy is zero and the kinetic energy has its maximum value. The system's inertia causes it to continue past the equilibrium position. The spring, which always acts to return the mass to $q = 0$, slows the mass until it comes to rest at the maximum displacement, where $|q| = A$ and $\dot{q} = 0$. Thus, the kinetic energy is zero at this position and the potential energy is a maximum, corresponding to the maximum spring deformation. The spring then pulls the mass back to $q = 0$, after which the process is repeated on the other side of $q = 0$. This vibration continues periodically, with frequency $\omega_{\text{nat}} = \sqrt{K/M}$, because there is no damping to dissipate the mechanical energy.

The development thus far is based on the stiffness coefficient K being positive, but that is not always the case. Recall that the basic definition of K for a one-degree-of-freedom system is

$$K = \left(\frac{\partial^2 V}{\partial q^2} \right)_{q=0} \tag{2.2.11}$$

Consider the potential energy for a ball at the top of a rounded hill, such as the one depicted in Figure 2.7. Clearly, this is a case of unstable static equilibrium, for any disturbance will cause the ball to roll away from the top of the hill. If we use the elevation z as the generalized coordinate, it is evident that the potential energy is a maximum at the top position, so we have $\partial V/\partial z = 0$ and $\partial^2 V/\partial z^2 < 0$. Thus, the stiffness K is negative. In the next section we will examine the effects of damping. Because such forces oppose velocity, they cannot cause the system to return to its equilibrium position. Consequently, we conclude that, in general,

If K for a one-degree-of-freedom system is negative, then the equilibrium position is unstable.

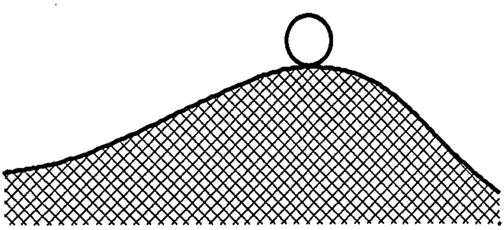


FIGURE 2.7 A ball at the top of a hill—a case of unstable static equilibrium.

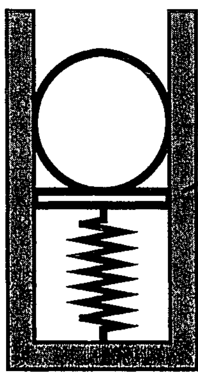
It is instructive to observe that if $K < 0$, then the homogeneous solution to the equation of motion is

$$q = B_1 \exp\left(\sqrt{\frac{-K}{M}}t\right) + B_2 \exp\left(-\sqrt{\frac{-K}{M}}t\right) \tag{2.2.12}$$

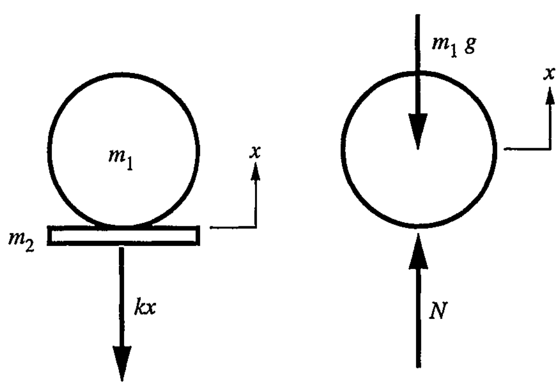
Thus, one of the solutions grows with increasing time. We say that this is a *divergence instability*. The small displacement approximations leading to linearized equations of motion are valid only for a very short time when a system is unstable. One indication of this loss of validity is the fact that the response in eq. (2.2.12) does not conserve energy. We will encounter another type of instability in Chapter 11, where we study time-dependent systems.

EXAMPLE 2.5 SEE REVISED VERSION

The 200 gram sphere is not attached to the 50 gram piston. The stiffness of the spring is 1200 N/m. The spring is held 80 mm below the static equilibrium position, and then released. Determine the position of the piston above the equilibrium position, and the corresponding elapsed time, at which the sphere ceases to be in contact with the piston.



Solution This exercise is intended to bring out the way in which static and dynamic forces might occur in a study, as well as to highlight interpretation of harmonic response. We begin by drawing two free body diagrams. The first, which we will use to derive the equation of motion, considers the sphere and the platform as a system, so that the contact force exerted between these bodies is an internal force that is not considered.



We consider the system to move away from its static equilibrium position a distance x upward, which serves as the generalized coordinate. Hence, the spring force acts downward. The gravity force has a static effect, which is irrelevant to the equation of motion because x is measured relative the static equilibrium position. Hence, the equation of motion resulting from the first free body diagram is

$$m\ddot{x} + kx = 0, \quad m = m_1 + m_2 = 0.25 \text{ kg}$$

The second free body diagram isolates the ball because we seek the conditions under which the contact force N applied by the platform becomes zero. We show the gravity force in this free body diagram because the weight affects the normal force N, and the condition we seek is $N = 0$. From the second free body diagram, we find that

$$N - m_1 g = m_1 \ddot{x}$$

Hence, $N = 0$ is marked by $\ddot{x} = -g$.

The idea now is to find the response corresponding to the given initial conditions, and then to determine when the acceleration condition $\ddot{x} = -g$ occurs. We write the equation of motion as

$$\ddot{x} + \omega_{\text{nat}}^2 x = 0$$

where the natural frequency is

$$\omega_{\text{nat}} = \left(\frac{k}{m}\right)^{1/2} = 69.28 \text{ rad/s}$$

The given initial conditions indicate that $x = -0.08$ m and $\dot{x} = 0$ at $t = 0$. The response matching these initial conditions is

$$x = -0.08\cos(\omega_{\text{nat}} t)$$

We seek the value of t_1 for which $\ddot{x} = -g$, but the equation of motion states that $\ddot{x} = -\omega_{\text{nat}}^2 x$, so we seek the condition for which $\omega_{\text{nat}}^2 x = g$, or

$$-0.08\,\omega_{\text{nat}}^2 \cos(\omega_{\text{nat}} t_1) = g$$

The solution of this relation is

$$\begin{aligned} t_1 &= \frac{1}{\omega_{\text{nat}}}\cos^{-1}\left(-\frac{g}{0.08\,\omega_{\text{nat}}^2}\right) = \frac{1}{69.28}\cos^{-1}\left(-\frac{9.807}{0.08 \times 69.28^2}\right) \\ &= 0.014434\cos^{-1}(-0.02554) \end{aligned}$$

We select for the inverse cosine the smallest angle that gives $t_1 > 0$, $\cos^{-1}(-0.02554) = 1.5963$, which leads to

$$t_1 = 0.02304 \text{ s}$$

2.2.2 Underdamped Systems

If the amount of damping is small, the time scale over which an appreciable amount of energy is lost will be large relative to the natural period of free vibration, $\tau_{\text{nat}} = 2\pi/\omega_{\text{nat}}$. In that case it might be acceptable to ignore dissipation effects for a short time interval. However, any amount of damping will eventually quiet the system, as we will see here.

When damping is present, we use the natural frequency ω_{nat} to rewrite the equation of motion as

$$\ddot{q} + 2\zeta\omega_{\text{nat}}\dot{q} + \omega_{\text{nat}}^2 q = 0 \tag{2.2.13}$$

A comparison of this form with eq. (2.2.1) shows that the parameter ζ is

$$\boxed{\zeta = \frac{C}{2\omega_{\text{nat}} M} \equiv \frac{C}{2\sqrt{KM}}} \tag{2.2.14}$$

The corresponding characteristic equation is

$$\boxed{\lambda^2 + 2\zeta\omega_{\text{nat}}\lambda + \omega_{\text{nat}}^2 = 0} \tag{2.2.15}$$

The above characteristic equation is quadratic, so its roots are either both real, or complex conjugates, given by

$$\lambda = -\zeta\omega_{\text{nat}} \pm \omega_{\text{nat}}\sqrt{\zeta^2 - 1} \tag{2.2.16}$$

It is obvious that $\zeta = 1$ leads to a transition in the nature of the roots, so we refer to ζ as the *critical damping ratio.* We begin by considering the case $0 < \zeta < 1$. Because the damping is less than critical in this case, the system is said to be *underdamped.* (The undamped model we considered previously could be treated as a special case of an underdamped system, if we included $\zeta = 0$.)

The discriminant $\zeta^2 - 1$ of the characteristic equation is negative, so the roots are complex conjugates. We write them as

$$\boxed{\lambda = -\zeta\omega_{\text{nat}} \pm i\omega_{\text{d}}} \tag{2.2.17}$$

where ω_{d} is the *damped natural frequency,*

$$\boxed{\omega_{\text{d}} = \omega_{\text{nat}}\sqrt{1 - \zeta^2}} \tag{2.2.18}$$

As we did for undamped systems, we associate a different coefficient with the exponential function for each characteristic root. Because the exponential of a sum is the product of the individual exponentials, we factor out $\exp(-\zeta\omega_{\text{nat}}t)$ from each solution, which leads to

$$q = \exp(-\zeta\omega_{\text{nat}}t)[B_1\exp(i\omega_{\text{d}}t) + B_2\exp(i\omega_{\text{d}}t)] \tag{2.2.19}$$

Aside from the $\exp(-\zeta\omega_{\text{nat}}t)$ factor and the frequency being ω_{d} rather than ω_{nat}, this expression is just like eq. (2.2.5) for undamped free vibration. We follow that development to enforce the requirement that q is real. Depending on whether we write the bracketed term in the form of eq. (2.2.6), eq. (2.2.7), or eq. (2.2.10), the solution for q may be expressed as

$$\begin{aligned} q &= \exp(-\zeta\omega_{\text{nat}}t)\text{Re}[\hat{A}\exp(i\omega_{\text{d}}t)] \\ &= \exp(-\zeta\omega_{\text{nat}}t)[c_1\cos(\omega_{\text{d}}t) + c_2\sin(\omega_{\text{d}}t)], \quad \hat{A} = c_1 - ic_2 \\ &= A\exp(-\zeta\omega_{\text{nat}}t)\text{Re}\{\exp[i(\omega_{\text{d}}t - \phi)]\}, \quad \hat{A} = A\exp(-i\phi) \end{aligned} \tag{2.2.20}$$

The unknown coefficients are set by the initial conditions, as they are for an undamped response. We use the second of the above representations to evaluate q and $\dot{q}$ at $t = 0$, which leads to

$$q_0 = c_1, \qquad \dot{q}_0 = -\zeta\omega_{\text{nat}}c_1 + \omega_{\text{d}}c_2 \quad \Rightarrow \quad c_2 = \frac{\dot{q}_0 + \zeta\omega_{\text{nat}}q_0}{\omega_{\text{d}}} \tag{2.2.21}$$

The real form of the response is therefore

$$\boxed{q = \exp(-\zeta\omega_{\text{nat}}t)\left[q_0\cos(\omega_{\text{d}}t) + \frac{\dot{q}_0 + \zeta\omega_{\text{nat}}q_0}{\omega_{\text{d}}}\sin(\omega_{\text{d}}t)\right]} \tag{2.2.22}$$

Although the second form in eqs. (2.2.20) is the one we use to satisfy the initial conditions, the last form is most useful for discussions. One way of picturing the last of eqs. (2.2.20) is to consider the factor $A\exp(-\zeta\omega_{\text{nat}}t)$ as the decaying amplitude of a harmonic function at frequency ω_{d}. In Figure 2.3, we represented a harmonic function as a rotating vector in the complex plane. From this viewpoint the underdamped response appears as a vector that rotates counterclockwise at angular speed ω_{d}, with an amplitude that decays exponentially. Thus, the tip of the vector follows an inward spiral, as shown in Figure 2.8. Note that in the figure $\omega_{\text{d}}t$ measures the angle relative to the orientation at $t = 0$, whereas the angle $\omega_{\text{d}}t - \phi$ is the phase variable of the cosine term above.

The more conventional way of viewing the response is to plot q as a function of t. In advance of plotting a typical response, we may anticipate the qualitative aspects by recognizing that $\cos(\omega_{\text{d}}t - \phi)$ oscillates between -1 and 1. Hence, the largest possible positive value of q at any instant is $A\exp(-\zeta\omega_{\text{nat}}t)$, and the largest negative value is $-A\exp(-\zeta\omega_{\text{nat}}t)$. The curves $q = \pm A\exp(-\zeta\omega_{\text{nat}}t)$ are the *envelope* of the underdamped response. Within this envelope the signal oscillates at the damped natural frequency. The elapsed time between zeroes is one-half the *damped period* τ_{d},

$$\boxed{\tau_{\text{d}} = 2\pi/\omega_{\text{d}}} \tag{2.2.23}$$

Figure 2.9 indicates that τ_{d} also gives the interval over which the positive and negative peak values of q occur, which is an aspect that we shall prove because it has important implications for system identification. However, this repetition should not be taken to imply that the underdamped response is periodic, because a nonzero value $|q(t + \tau_{\text{d}})|$ is always smaller than $|q(t)|$.

We begin to investigate timing issues by observing that the $q(t)$ curve tangentially intersects the envelope whenever $|\cos(\omega_{\text{d}}t - \phi)| = 1$, so the interval between adjacent positive or negative intersections is τ_{d}. Because the zeroes occur when $|\cos(\omega_{\text{d}}t - \phi)| = 0$, intersections with the envelope occur at intervals that are separated from the zeroes by $\tau_{\text{d}}/4$. Next, we note that the slope of the envelope is negative,

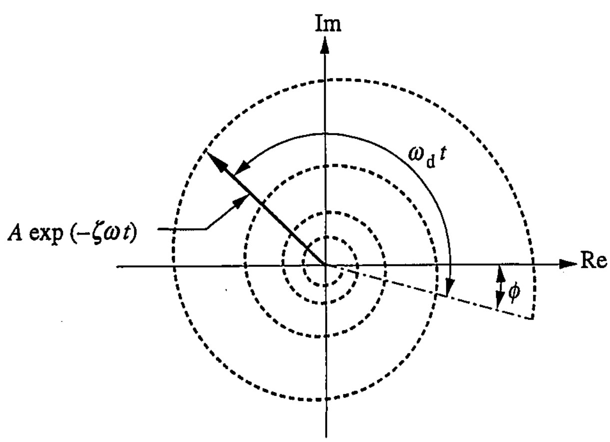


FIGURE 2.8 Underdamped response as a rotating vector in the complex plane.

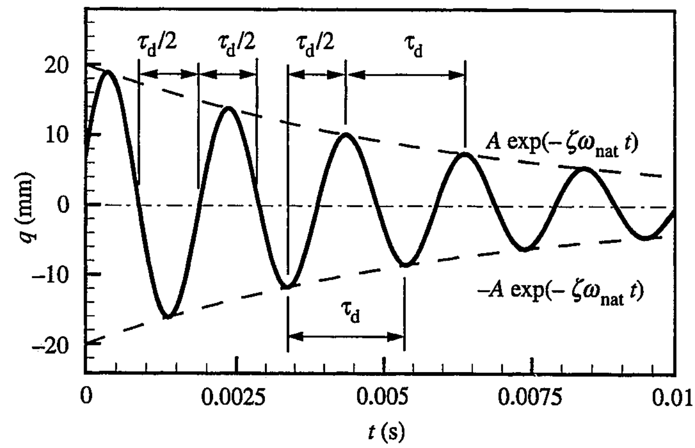


FIGURE 2.9 Typical underdamped response, $\omega_{nat} = 1000\pi$ rad/s, $\zeta = 0.05$.

so the intersections with the envelope occur later than the instant at which $|q|$ has a maximum value. In other words, *maxima or minima of q(t) do not occur midway between the occurrence of zeroes.* (This aspect has been emphasized because failure to recognize it is a common mistake for novices.) To find the instants at which max $|q|$ occur, we must determine when $\dot{q} = 0$. We obtain an expression for $\dot{q}$ by differentiating eq. (2.2.22), which we simplify by using the definition of ω_d,

$$\dot{q} = \exp(-\zeta\omega_{nat}t)\left\{\dot{q}_0\cos(\omega_d t) - \left[\omega_d q_0 + \zeta\omega_{nat}\left(\frac{\dot{q}_0 + \zeta\omega_{nat}q_0}{\omega_d}\right)\right]\sin(\omega_d t)\right\} \tag{2.2.24}$$

$$\equiv \exp(-\zeta\omega_{nat}t)\left\{\dot{q}_0\cos(\omega_d t) - \frac{\omega_{nat}}{\omega_d}(\omega_{nat}q_0 + \zeta\dot{q}_0)\sin(\omega_d t)\right\}$$

Let us denote as $t_j, j = 1, 2, \ldots$, the roots of $\dot{q} = 0$ for which q has a maximum positive value, while $t_{j+1/2}$ denotes instants at which $\dot{q} = 0$ and q has a maximum negative value. Because τ_d is the period of both trigonometric functions appearing in eq. (2.2.24), increasing t from t_j to $t_j + \tau_d$ leads to the same value of the sine and cosine functions, so we have $t_{j+1} = t_j + \tau_d$. Furthermore, if we increase t from t_j to $t_j + \tau_d/2$, the values of both functions are the negative of their values at $t = t_j$. This leads us to the conclusion that $t_{j+1/2} = t_j + \tau_d/2$, from which it follows that $t_{j+1/2} = t_{j-1/2} + \tau_d$. In summary, we find that

- Zeroes of the underdamped response occur at intervals of $\tau_d/2$.
- Adjacent maxima and minima are separated by intervals of $\tau_d/2$.
- Maxima and minima occur slightly earlier than $\tau_d/4$ following the previous zero.

These features are displayed in Figure 2.9.

The repetitive nature of the peaks leads us to a simple equation by which the critical damping ratio of a system may be determined from measurements of $q(t)$. Let us denote the maxima as $x_j \equiv q(t_j)$. We do not need to know the actual value of t_j, which would require solving eq. (2.2.24). Rather we only require a comparison of x_j and x_{j+1},

$$x_j = q(t_j) = A\exp(-\zeta\omega_{\text{nat}}t_j)\cos(\omega_d t_j - \phi)$$

$$\begin{aligned} x_{j+1} &= q(t_j + \tau_d) = A\exp\left[-\zeta\omega_{\text{nat}}\left(t_j + \frac{2\pi}{\omega_d}\right)\right]\cos\left[\omega_d\left(t_j + \frac{2\pi}{\omega_d}\right) - \phi\right] \\ &= \exp\left(-\frac{2\pi\zeta\omega_{\text{nat}}}{\omega_d}\right)A\exp(-\zeta\omega_{\text{nat}}t_j)\cos[\omega_d t_j - \phi] \\ &= x_j\exp\left(-\frac{2\pi\zeta\omega_{\text{nat}}}{\omega_d}\right) \equiv x_j\exp\left[-\frac{2\pi\zeta}{(1-\zeta^2)^{1/2}}\right] \end{aligned} \tag{2.2.25}$$

A similar analysis applied to a comparison of the minima, which we denote as $x_{j+1/2} \equiv |q(t_{j+1/2})|$, would show that

$$x_{j+1/2} = x_j\exp\left[-\frac{\pi\zeta}{(1-\zeta^2)^{1/2}}\right] \tag{2.2.26}$$

Suppose we have measured $q(t)$, which means that we know the values of several x_j. To find the value of ζ associated with the response, we form the ratio of the successive maxima, which leads to the *log decrement* δ, where

$$\boxed{\delta = \ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\zeta}{(1-\zeta^2)^{1/2}}} \tag{2.2.27}$$

We may determine the critical damping ratio from δ by solving this expression, which yields

$$\boxed{\zeta = \frac{\delta}{(4\pi^2 + \delta^2)^{1/2}}} \tag{2.2.28}$$

Many systems are said to be *lightly damped*, which means that their ratio of critical damping is small. In such systems the log decrement will also be a small value, so we may approximate the above relation as $\zeta \approx \delta/2\pi$, which is quite usable for $\zeta < 0.1$.

The smallness of δ in a lightly damped system brings up the issue of experimental error and its influence on the resulting value of ζ. A small δ leads to x_{j+1} being only slightly smaller than x_j. If this difference is of the order of magnitude of the measurement error, then the value of ζ derived from eqs. (2.2.27) and (2.2.28) will be quite inaccurate. We may improve the evaluation by comparing the maxima after a large number of cycles N. In view of the exponential decay of the successive values of q_j, we have

$$\boxed{\delta = \frac{1}{N}\ln\left(\frac{x_j}{x_{j+N}}\right)} \tag{2.2.29}$$

As a closure to this discussion, we should note that one could carry out the same evaluations equally well by using successive minimum values $x_{j+1/2}$.

Let us now consider a situation where $\zeta \ll 1$, in order to identify when it might be sufficient to use an undamped model to study free vibration of a system. In that case we may use power series in ζ to obtain alternative forms. Equation (2.2.18) indicates that the damped and undamped natural frequencies for small ζ are approximately related by $\omega_d \approx \omega_{nat}(1 - \frac{1}{2}\zeta^2)$. In other words light damping has a second-order effect on the observed oscillation rate. In contrast, when we expand the exponential in eq. (2.2.25), we find that $x_{j+1} \approx x_j(1 - 2\pi\zeta)$, which is a first-order effect. Thus, when we observe the response of a very lightly damped system, the fact that x_{j+1} is smaller than x_j is more observable than the fact that ω_d is smaller than ω_{nat}. It follows that we may use an undamped model to represent free vibration of a lightly damped system, provided that we limit the observation interval to be shorter than the number of damped periods required to obtain a reasonably precise value of δ from eq. (2.2.29).

For a different perspective on the significance of the log decrement, let us consider the mechanical energy that is dissipated in an underdamped free vibration. Because peak values x_j are defined to be maxima, at which $\dot{q} = 0$, it follows that the mechanical energy $E \equiv T + V$ corresponding to the peak values is solely stored as potential energy, so we have

$$E_j = \tfrac{1}{2}kx_j^2, \quad E_{j+1} = \tfrac{1}{2}kx_{j+1}^2 \tag{2.2.30}$$

From the definition of the log decrement, eq. (2.2.27), we have $x_{j+1} = x_j \exp(-\delta)$. The energy dissipated from one peak to the next is $E_j - E_{j+1}$, so we find that the fraction of the mechanical energy dissipated in a cycle is

$$\frac{E_j - E_{j+1}}{E_j} = 1 - \exp(-2\delta) \approx 2\delta \quad \text{if } \delta \ll 1 \tag{2.2.31}$$

In other words, in each damped cycle the fraction of mechanical energy lost to damping is approximately 2δ. Obviously, $1 - 2\delta$ is the fraction of energy that remains, so linear damping can never bring the system fully to rest in a finite amount of time.

Before we proceed to the case of an overdamped system, it is useful to examine the influence of increased damping in the underdamped case. Figure 2.10 shows responses for four large values of ζ; the natural frequency and initial conditions are the same as those for Figure 2.9. It is evident that the attenuation rate increases drastically as damping is increased, but the damped period is affected much less. (Recall that the effect of ζ on τ_d is second order.) The response for $\zeta = 0.8$ is interesting because it shows that there

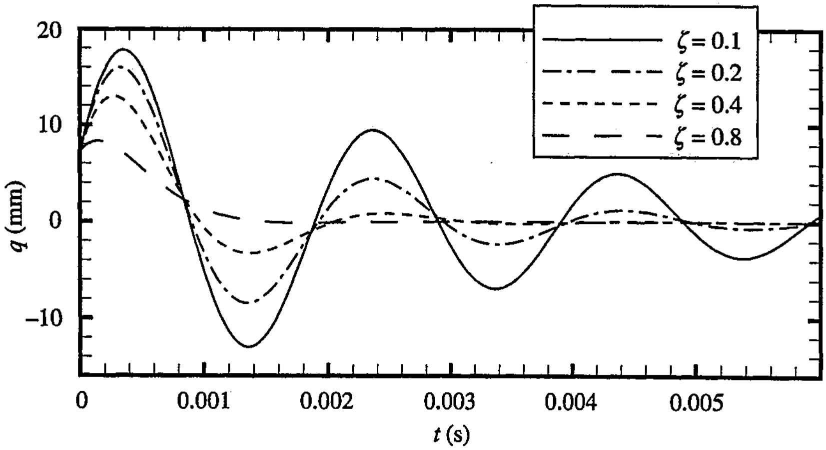


FIGURE 2.10 Effect of increasing damping for large damping ratios below critical, $\omega_{nat} = 1000\pi$ rad/s.

is almost no oscillation. Essentially what happens is that the *time constant* $1/(\zeta\omega_{\text{nat}})$ for the envelope, which is the time required for the envelope to decay by a factor of $1/e$, becomes comparable to the damped period, so that there is very little response at the end of a single cycle. Critical damping represents the limit of this trend. As we will see, overdamped and critically damped systems do not show oscillations in their free response.

EXAMPLE 2.6

A one-degree-of-freedom system has mass $M = 4$ kg and spring stiffness $K = 16(10^4)$ N/m. The system is known to be underdamped, but the damping constant C is unknown. The system is released from rest at $t = 0$ with an initial value of the generalized coordinate $q(t = 0) = -40$ mm. It is observed that the largest positive value of q in the free vibration occurs at $t = 0.02$ s. Determine (a) the value of q at $t = 0.02$ s, (b) the value of C, (c) the earliest instant at which $q = 0$ after the system is released, and the velocity $\dot{q}$ at that instant.

Solution This exercise will highlight the relationship between the damped period, critical damping ratio, and features of the response as a function of time. We begin by placing the given aspects of the response in the context of the standard properties of underdamped system response. It is stated that at $t = 0$, $q_0 = -0.040$ m. We also know that at this instant $\dot{q}_0 = 0$, so a plot of q as a function of t has a horizontal slope at $t = 0$. Because q_0 is negative at this instant, it must be that $t = 0$ corresponds to a minimum value of q. Furthermore, $t = 0.02$ s is the time at which the first maximum occurs, because successive maxima decrease in magnitude and $t = 0.02$ s is stated to be the largest. We saw in Figure 2.9 that the instants when maxima occur are equally spaced between instants when minima occur. It follows that the interval from $t = 0$ to $t = 0.02$ s is one-half the damped period, so we have $\tau_d = (2)\,0.02$ s and

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{0.04} = 157.08 \text{ rad/s}$$

We determine the undamped natural frequency from the given system parameters,

$$\omega_{\text{nat}} = \sqrt{\frac{K}{M}} = 200 \text{ rad/s}$$

The relation between the damped and undamped natural frequencies then yields

$$(1 - \zeta^2)^{1/2} = \frac{\omega_d}{\omega_{\text{nat}}} = 0.7854$$

From this, we find the critical damping ratio, which leads directly to the damping constant C,

$$\zeta = (1 - 0.7854^2)^{1/2} = 0.6190$$

$$\frac{C}{M} = 2\zeta\omega_{\text{nat}} \quad \Rightarrow \quad C = 4(2)(0.6190)(200) = 990.4 \text{ N-s/m}$$

Knowledge of the critical damping ratio also enables us to evaluate the log decrement, from which we may compute q at $t = 0.02$ s. First, we have

$$\delta = \frac{2\pi\zeta}{(1 - \zeta^2)^{1/2}} = 4.952$$

Because the interval from $t = 0$ to $t = 0.02$ s constitutes a half-cycle, we may employ eq. (2.2.29) with $N = 1/2$ and the minimum at the start of the interval set as $x_0 = q_0$,

$$\delta = \frac{1}{(1/2)}\ln\left(\left|\frac{q_0}{x_{1/2}}\right|\right) \quad \Rightarrow \quad |x_{1/2}| = |q_0|\exp\left(-\frac{\delta}{2}\right) = 3.363 \text{ mm}$$

The remaining properties to be evaluated pertain to the earliest instant at which $q = 0$. This condition does not occur at one-quarter of a damped period after the minimum. We identify the condition by examining the response as a function of time. The underdamped system response in eq. (2.2.22) corresponding to $\dot{q}_0 = 0$ is

$$q = q_0 \exp(-\zeta\omega_{\mathrm{nat}} t)\left[\cos(\omega_{\mathrm{d}} t) + \frac{\zeta\omega_{\mathrm{nat}}}{\omega_{\mathrm{d}}}\sin(\omega_{\mathrm{a}} t)\right]$$

To determine the instant t' at which $q = 0$, we set

$$\cos(\omega_{\mathrm{d}} t') + \frac{\zeta\omega_{\mathrm{nat}}}{\omega_{\mathrm{d}}}\sin(\omega_{\mathrm{d}} t') = 0$$

from which we obtain

$$t' = \frac{1}{\omega_{\mathrm{d}}}\tan^{-1}\left(-\frac{\omega_{\mathrm{d}}}{\zeta\omega_{\mathrm{nat}}}\right) \equiv \frac{1}{\omega_{\mathrm{d}}}\tan^{-1}\left[-\frac{(1-\zeta^2)^{1/2}}{\zeta}\right] = 0.01425$$

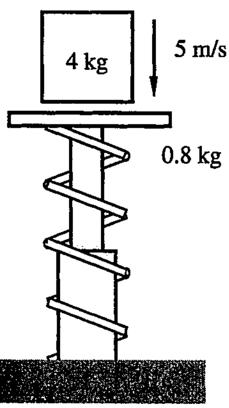
Note that the argument of the arctangent is negative. A calculator or computer program is likely to return a negative angle for the function, so the computed value of the arctangent must be increased by π in order to obtain a positive value for t'. The velocity at t' is readily determined by differentiating the solution for q, which gives

$$\dot{q}' = q_0 \exp(-\zeta\omega_{\mathrm{nat}} t')\left\{-\zeta\omega_{\mathrm{nat}}\left[\cos(\omega_{\mathrm{d}} t') + \frac{\zeta\omega_{\mathrm{nat}}}{\omega_{\mathrm{d}}}\sin(\omega_{\mathrm{d}} t')\right]\right.$$

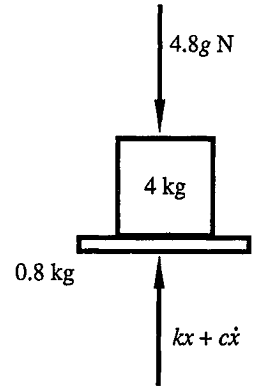
$$\left. + \omega_{\mathrm{d}}\left[-\sin(\omega_{\mathrm{d}} t') + \frac{\zeta\omega_{nat}}{\omega_{\mathrm{d}}}\cos(\omega_{\mathrm{d}} t')\right]\right\} = 1445.5 \text{ mm/s}$$

EXAMPLE 2.7

A 0.8 kg platform is supported by a shock absorber. The stiffness of the spring is 600 N/m and the dashpot constant is 20 N-s/m. The platform is at rest in its static equilibrium position when a 4 kg package that is falling at 5 m/s impacts against it. The collision is perfectly plastic, so the platform and package move together in the subsequent motion. Determine the response after impact.



Solution A primary objective of this exercise is to illustrate the process of satisfying initial conditions. It will also emphasize the interplay between the static equilibrium position and the static effect of gravity. We begin by drawing a free body diagram of the platform and the package, which is the vibratory system following the impact.



It is stated that the platform was in static equilibrium prior to the impact, but the addition of the package changes the static equilibrium position, because the weight supported by the spring is increased. In order to avoid any ambiguity as to which equilibrium position we are using, we shall deviate here from our usual practice by defining the generalized coordinate to be the downward displacement x measured from the location at which the spring is undeformed. Correspondingly, we must include the static gravitational force in the free body diagram. Thus, the equation of motion is

$$\sum F = mg - kx - c\dot{x} = m\ddot{x} \quad \Rightarrow \quad \ddot{x} + 2\zeta\omega_{\text{nat}}\dot{x} + \omega_{\text{nat}}^2 x = g$$

where $m = 4.8$ kg, $k = 600$ N/m, and $c = 20$ N-s/m. This corresponds to

$$\omega_{\text{nat}} = \left(\frac{k}{m}\right)^{1/2} = 11.1803 \text{ rad/s}$$

$$\zeta = \frac{c}{2(km)^{1/2}} = 0.1864$$

This last parameter is especially important because it indicates that the system is indeed underdamped.

Evaluation of the response requires initial conditions. We use the fact that the momentum of a system is conserved in a collision to determine the initial velocity. Prior to the impact, the package was falling with a speed of 5 m/s, and immediately after the impact the package and the platform both have speed v_0. Equating the momentum immediately before and after the collision leads to

$$4.8v_0 = 4(5) \quad \Rightarrow \quad v_0 = 4.1667 \text{ m/s}$$

The initial displacement is governed by the equations of static equilibrium. For $t < 0$, the system was in equilibrium under the weight of the platform, so the initial force in the spring equals the weight of the platform, $kx_0 = 0.8g$ N, from which we obtain

$$x_0 = \frac{0.8(9.807)}{600} = 0.013076 \text{ m}$$

The equation of motion has a constant term to the right of the equality sign, so we must add a constant particular solution to the complementary solution usually associated with free vibration. (We will review this aspect of solving equations of motion in a later section.) The general solution that results is

$$x = \frac{g}{\omega_{\text{nat}}^2} + \exp(-\zeta\omega_{\text{nat}}t)[c_1\cos(\omega_{\text{d}}t) + c_2\sin(\omega_{\text{d}}t)]$$

where

$$\omega_{\text{d}} = (1 - \zeta^2)^{1/2}\omega_{\text{nat}} = 10.984 \text{ rad/s}$$

We determine the coefficients c_1 and c_2 by satisfying the initial conditions. For displacement, we have

$$0.013076 = \frac{9.807}{11.1803^2} + c_1$$

while the initial velocity condition is

$$4.1667 = -\zeta\omega_{\text{nat}}c_1 + \omega_d c_2$$

Thus, we find

$$c_1 = -0.06537, \qquad c_2 = 0.3917$$

The response is

$$x = 0.7854 + \exp(-2.083t)[-0.06537\cos(10.984t) + 0.3917\sin(10.984t)] \text{ m}$$

2.2.3 Overdamped Systems

The nature of the solution of the equation of motion is dictated by the characteristic roots in eq. (2.2.16). We have already seen that these roots are complex conjugates when $\zeta < 1$. Let us consider a graph in which we plot the roots in the complex plane corresponding to different values of ζ with ω_{nat} held fixed. As shown in Figure 2.11, between $\zeta = 0$ and $\zeta = 1$, the roots are complex conjugates that lie on a circle whose radius is ω_{nat}. At $\zeta = 1$, the two roots merge. Increasing the critical damping ratio beyond $\zeta = 1$ causes the roots to split up, migrating in opposite directions along the real axis. As $\zeta \longrightarrow \infty$, one root approaches zero, and the other becomes infinite.

When $\zeta > 1$, we say that the system is *overdamped.* Both characteristic roots are negative, which we emphasize by indicating the minus sign explicitly. Thus, the roots of the characteristic eq. (2.2.15) for $\zeta > 1$ are written as

$$\boxed{\begin{gathered}\lambda = -\lambda_1, -\lambda_2 \\ \lambda_1 = \zeta\omega_{\text{nat}} - \omega_{\text{nat}}\sqrt{\zeta^2 - 1}, \qquad \lambda_2 = \zeta\omega_{\text{nat}} + \omega_{\text{nat}}\sqrt{\zeta^2 - 1}\end{gathered}} \tag{2.2.32}$$

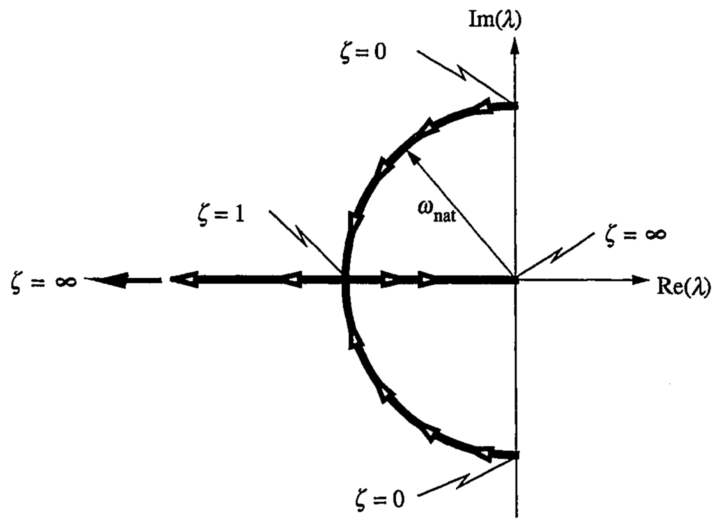


FIGURE 2.11 Characteristic roots in the complex plane as a function of ζ.